

Fractional Calculus Involving Multivariable H-Function

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Abstract:

In this Paper, we use fractional differential operators $D_{k,\alpha,x}^n$ and ${}_x D_x^\mu$ to derive a number of key formulas of multivariable H-function. Which involve a product of two multi variables H-Function. It is further shown that, each of these formulas yield interesting new formulas for certain multivariable hyper geometric H-function

Keywords - Fractional differential Operator, Hyper geometric -function.

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Definition and Introduction

We use the fractional derivative operator defined in the following manner [5],

$$D_{k,\alpha,x}^n(x^\mu) = \prod_{r=0}^{n-1} \left[\frac{\sqrt{\mu + rk + 1}}{\sqrt{\mu + rk - \alpha + 1}} \right] x^{\mu+nk} \quad (1.1)$$

Where $\alpha \neq \mu + 1$ and α and k are not necessarily integers.

We use the binomial expansion in the following manner,

$$(ax^\mu + b)^\lambda = b^\lambda \sum_{l=0}^{\infty} \binom{\lambda}{l} \left(\frac{ax^\mu}{b} \right)^l \quad \text{where } \left[\frac{ax^\mu}{b} \right] < 1 \quad (1.2)$$

MAIN RESULT

Theorem- 1 Fractional Differential Operator Involving Multivariable H-Function

$$\begin{aligned} &\Rightarrow D_x^\mu \{ x^k (x^{v_1} + a)^\lambda (b - x^{v_2})^\delta H[z_1 x^{p_1} \dots z_r x^{p_r}] \} \\ &\Rightarrow a^\lambda b^\delta x^{k-\mu} \sum_{l,m=0}^{\infty} \left(\frac{x^{v_1}}{a} \right)^l \left(\frac{x^{v_2}}{b} \right)^m \binom{\lambda}{l} \binom{\delta}{m} H_{p+1,q+1,p_1,q_1,\dots,p_r,q_r}^{0,n+1;m_1,n_1,\dots,m_r,n_r} \\ &\quad \left[\begin{matrix} z_1 x^{p_1} \\ \vdots \\ z_r x^{p_r} \end{matrix} \right]_{(-k+l v_1 - m v_2; p_1, \dots, p_r) (a; \alpha_j^1, \dots, \alpha_j^{(r)})_{1,p} ; (c_j^1 r_j^1)_{1,p_1} ; \dots ; (c_j^{(r)} r_j^{(r)})_{1,p_r}} \\ &\quad \left[\begin{matrix} z_1 x^{p_1} \\ \vdots \\ z_r x^{p_r} \end{matrix} \right]_{(-k+u-l v_1 - m v_2; p_1, \dots, p_r) (b; \beta_j^1, \dots, \beta_j^{(r)})_{1,q} ; (d_j^1 s_j^1)_{1,q_1} ; \dots ; (d_j^{(r)} s_j^{(r)})_{1,q_r}} \end{aligned} \quad (2.1)$$

Provided (in addition to the appropriate convergence and existence condition) that

$$\min \{ \nu_1, \nu_2, \rho_i, \sigma_i, \delta_i \} > 0 \quad (i = 1, \dots, r)$$

$$\max \left\{ \left| \arg \left(x^{\nu_1} / a \right) \right|, \left| \arg \left(x^{\nu_2} / b \right) \right| \right\} < \pi \quad \operatorname{Re}(k) + \sum_{i=1}^r \rho_i \xi_i > -1$$

Proof:- Now by Using Mellian Barnes Contour Integrals. Interpreting the above result of the definition of the H-function of several variables we get the required result

Theorem – 2 Fractional Differential Operators Involving Double Multivariable H-Function

$$\begin{aligned} & \Rightarrow D_x^\mu \{ x^k (x^{\nu_1} + a)^\lambda (b - x^{\nu_2})^\delta H[z_1 x^{\rho_1} \dots z_r x^{\rho_r}] H^* [\omega_1 x^{\lambda_1} \dots \omega_s x^{\lambda_s}] \} \\ & \Rightarrow a^\lambda b^\delta x^{k-\mu} \sum_{l,m,u=0}^{\infty} \binom{\lambda}{l} \binom{\delta}{m} \binom{\mu}{u} \left(\frac{x^{\nu_1}}{a} \right)^l \left(\frac{x^{\nu_2}}{b} \right)^m H_{p+1,q+1,\dots,p_r,q_r}^{0,n+1;m_1,n_1,\dots,m_r,n_r} \\ & \quad \times \left[\begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{matrix} \right]_{(-l\nu_1-m\nu_2;\rho_1,\dots,\rho_r) \left(a_j; \alpha_j^1, \dots, \alpha_j^{(r)} \right)_{1,p_1} \dots \left(c_j^1 r_j^1 \right)_{1,p_r} \left(c_j^{(r)} r_j^{(r)} \right)_{1,p_1}} \\ & \quad \times H_{p+1,q+1,\dots,p_r,q_r}^{0,n+1;m_1,n_1,\dots,m_r,n_r} \left[\begin{matrix} \omega_1 x^{\lambda_1} \\ \vdots \\ \omega_s x^{\lambda_s} \end{matrix} \right]_{(-k+\mu-u,\lambda_1,\dots,\lambda_s) \left(g_j; G_j^1, \dots, G_j^{(s)} \right)_{1,p_1} \left(u_j^1 U_j^1 \right)_{1,p_1} \dots \left(\mu_j^{(s)} U_j^{(s)} \right)_{1,p_s}} \\ & \quad \times H_{p+1,q+1,\dots,p_r,q_r}^{0,n+1;m_1,n_1,\dots,m_r,n_r} \left[\begin{matrix} \omega_1 x^{\lambda_1} \\ \vdots \\ \omega_s x^{\lambda_s} \end{matrix} \right]_{(-k+\mu-u,\lambda_1,\dots,\lambda_s) \left(h_j; H_j^1, \dots, H_j^{(s)} \right)_{1,q} \left(v_j^1 V_j^1 \right)_{1,q_1} \dots \left(v_j^{(s)} V_j^{(s)} \right)_{1,q_s}} \end{aligned} \quad (2.2)$$

Provided (in addition to the appropriate convergence and existence condition) that

$$\min \{ \nu_1, \nu_2, \rho_i, \sigma_i, \delta_i \} > 0 \quad (i = 1, \dots, r)$$

$$\max \left\{ \left| \arg \left(x^{\nu_1} / a \right) \right|, \left| \arg \left(x^{\nu_2} / b \right) \right| \right\} < \pi \quad \operatorname{Re}(k) + \sum_{i=1}^r \rho_i \xi_i > -1$$

Proof:- Now by Using Mellian Barnes Contour Integrals. Interpreting the above result of the definition of the H-function of several variables we get the required result

Theorem–3 Double Fractional Differential Operator Involving Multivariable H-Function

$$\Rightarrow D_x^\mu D_y^w \{ x^k y^\lambda (x^{\nu_1} + a)^\sigma (b - x^{\nu_2})^{-\delta} (y^{\nu_3} + c)^h (d - y^{\nu_4})^{-g} H[z_1 x^{\lambda_1} \dots z_r x^{\lambda_r}] \}$$

Provided (in addition to the appropriate convergence and existence conditions mentioned with (4.4.1))

$$= x^{k-\mu} y^{\lambda-w} a^\sigma b^{-\delta} c^h d^{-g} \sum_{l,m,r,s=0}^{\infty} \binom{\sigma}{l} \binom{\delta}{m} \binom{h}{r} \binom{g}{s} \left(\frac{x^{\nu_1}}{a} \right)^l \left(\frac{x^{\nu_2}}{b} \right)^m \left(\frac{y^{\nu_3}}{c} \right)^r \left(\frac{y^{\nu_4}}{d} \right)^s$$

$$H_{p+q+2, p_1, q_1, \dots, p_r, q_r}^{0, n+2, m_1, n_1, \dots, m_r, n_r} \times \left[\begin{array}{c} \left[z_1 x^{k_1} y^{\lambda_1} \right]_{(-k-lv_1-mv_2; k_1, \dots, k_r)(-\lambda-rv_3-sv_4; \lambda_1 \dots \lambda_r)} \\ \left[z_r x^{k_r} y^{\lambda_r} \right]_{(-k+\mu-lv_1-mv_2; k_1, \dots, k_r)(-\lambda+\mu-rv_3-sv_4; \lambda_1 \dots \lambda_r)} \\ \left(a_j, a_j^1 \dots a_j^{(r)} \right)_{1,p} \left(c_j^1 \gamma_j^1 \right)_{1,p_1} \dots \left(c_j^{(r)} \gamma_j^{(r)} \right)_{1,p_r} \\ \left(b_j, \beta_j^1 \dots \beta_j^{(r)} \right)_{1,q} \left(d_j^1 \delta_j^1 \right)_{1,q_1} \dots \left(d_j^{(r)} \delta_j^{(r)} \right)_{1,q_r} \end{array} \right] \quad (2.3)$$

Provided (in addition to the appropriate convergence and existence condition) that

$$\min \{v_1, v_2, \rho_i, \sigma_i, \delta_i\} > 0 \quad (i = 1, \dots, r)$$

$$\max \left\{ \left| \arg \left(x^{v_1} / a \right) \right|, \left| \arg \left(x^{v_2} / b \right) \right| \right\} < \pi \quad \operatorname{Re}(k) + \sum_{i=1}^r \rho_i \xi_i > -1$$

Proof:- Now by Using Mellian Barnes Contour Integrals .Interpreting the above result of the definition of the H-function of several variables we get the required result

Conclusion

In this paper, we use fractional differential operators and to derive a number of key formulas of multivariable H-function. It is further shown that ,each of these formulas yield interesting new formulas for certain multivariable hyper geometric function such as generalized Lauricella function (Srivastava-Daoust) and Lauricella hyper geometric function some of these application of the key formulas provide potentially useful generalization of known result in the theory of fractional calculus.

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