

Generalized Fractional Derivative Operators Involving Fox's H-Function with Applications

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Abstract

In the present paper, we introduce left and rightsided generalized fractional derivative operators and define a Fox's H-function. We derive explicit expressions for the Riemann–Liouville fractional-derivative operator acting on the Fox's H-function. Special cases of our results recover known fractional-derivative formulas, uniting them within a single, cohesive framework.

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1. Generalized Fractional Derivative Operators

For $\mu = [\operatorname{Re}(\gamma) + 1]$ and $x > 0$, where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$,

the generalized fractional derivative operators are defined [22] as follows:

$$\begin{aligned} D_{0+}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} f(x) &= (I_{0+}^{-\alpha_2, -\alpha_1, -\beta_2, -\beta_1, -\gamma} f)(x); \operatorname{Re}(\gamma) > 0 \\ &= \left(\frac{d}{dx} \right)^\mu (I_{0+}^{-\alpha_2, -\alpha_1, -\beta_2 + \mu, -\beta_1, -\gamma + \mu} f)(x). \end{aligned} \quad (1.1)$$

$$\begin{aligned} D_{0-}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} f(x) &= (I_{0-}^{-\alpha_2, -\alpha_1, -\beta_2, -\beta_1, -\gamma} f)(x); \operatorname{Re}(\gamma) > 0 \\ &= \left(-\frac{d}{dx} \right)^\mu (I_{0-}^{-\alpha_2, -\alpha_1, -\beta_2, -\beta_1 + \mu, -\gamma + \mu} f)(x). \end{aligned} \quad (1.2)$$

Where generalized fractional integral operators are defined as

$$(I_{0+}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} f)(x)$$

$$= \frac{x^{-\alpha_1}}{\Gamma(\gamma)} \int_0^x \frac{(x-t)^{\gamma-1}}{t^{\alpha_2}} F_3 \left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt$$

and

$$(I_{0-}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} f)(x)$$

$$= \frac{x^{-\alpha_2}}{\Gamma(\gamma)} \int_x^\infty \frac{(t-x)^{\gamma-1}}{t^{\alpha_1}} F_3 \left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) dt.$$

The following image formula for a power function under the generalized fractional integral operators is given[21] as follows:

$$(I_{0+}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} t^{\varepsilon-1})(x) = \Gamma \left[\begin{matrix} \varepsilon, & \varepsilon + \gamma - \alpha_1 - \alpha_2 - \beta_1, & \varepsilon - \alpha_2 + \beta_2 \\ \varepsilon + \beta_2, & \varepsilon + \gamma - \alpha_1 - \alpha_2, & \varepsilon + \gamma - \alpha_2 - \beta_1 \end{matrix} \right] x^{\varepsilon - \alpha_1 - \alpha_2 + \gamma - 1}, \quad (1.3)$$

where $\text{Re}(\varepsilon) > \max\{\text{Re}(\alpha_1 + \alpha_2 + \beta_1 - \delta), \text{Re}(\alpha_2 - \beta_2), 0\}$ and

$$(I_{0-}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} t^{-\varepsilon})(x) = \Gamma \left[\begin{matrix} \alpha_1 + \alpha_2 - \gamma + \varepsilon, & \alpha_1 + \beta_2 - \gamma + \varepsilon, & -\beta_1 + \varepsilon \\ \varepsilon, & \alpha_1 + \alpha_2 + \beta_2 - \gamma + \varepsilon, & \alpha_1 - \beta_1 + \varepsilon \end{matrix} \right] x^{-\varepsilon - \alpha_1 - \alpha_2 + \gamma}, \quad (1.4)$$

where $\text{Re}(\varepsilon) < 1 + \min\{\text{Re}(-\beta_1), \text{Re}(\alpha_1 + \beta_2 - \gamma), \text{Re}(\alpha_1 + \alpha_2 - \gamma)\}$ and

$$\Gamma \left[\begin{matrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \end{matrix} \right] = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}. \quad (1.5)$$

Let $f_1(x) = \sum_{k=0}^\infty C_k x^k$ and $f_2(x) = \sum_{k=0}^\infty D_k x^k$ be two analytic functions with their radii of convergence R_{f_1} and R_{f_2} , respectively. Then their Hadamard product is

given by the following power series:

$$f_1 * f_2(x) = f_2 * f_1(x) = \sum_{k=0}^\infty C_k D_k x^k; \quad (|x| < R), \quad (1.6)$$

Where $R_c \geq R_{f_1} \cdot R_{f_2}$ is the radius of convergence of the composite series.

2. Generalized Fox's H-function

A lot of research work has been recently come up on the study and development of a function that is more general than Fox's H -function, known as H -function.

Charles Fox defined Fox H in 1961 [4], and means of the following Mellin-Barnes type of contour integral is presented.

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n} \left\{ z \left| \begin{matrix} (a_i, A_i)_1^p \\ (b_j, B_j)_1^q \end{matrix} \right. \right\} = \frac{1}{2\pi i} \int_L H_{p,q}^{m,n}[z] z^{-s} ds, z \neq 0 \quad (2.1)$$

Where L is suitable contour, $0 \leq m \leq q, 0 \leq n \leq p, A_i > 0, B_j > 0$ and $a_i, b_j \in \mathbb{C}; i=1, 2, \dots, p; j=1, 2, \dots, q, A_i(b_j + l) \neq B_j(a_i - l' - 1), l, l' = 0, 1, 2, \dots$

$$H_{p,q}^{m,n}[z] = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{i=1}^n \Gamma(1 - a_i - A_i s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \prod_{i=n+1}^p \Gamma(1 - a_i - A_i s)} \quad (2.2)$$

Contour separating the points

$$s = \left(\frac{b_j + 1}{B_j} \right); (l = 0, 1, 2, \dots, j = 1, 2, \dots, q)$$

Which are the Poles of $\Gamma(b_j - B_j s); (j=1 \dots q)$, from the points

$$s = \left(\frac{a_i - l' - 1}{A_i} \right); (l' = 0, 1, 2, \dots, i = 1, 2, \dots, p)$$

Which are the poles of this $\Gamma(1 - a_i - A_i s); i = 1 \dots p$

$$H_{p,q}^{m,n}[z] = \sum_{h=1}^m \sum_{r=1}^{\infty} \frac{\prod_{j=1}^m \Gamma(b_j + B_j \xi_{h,r}) \prod_{i=1}^n \Gamma(1 - a_i - A_i \xi_{h,r})}{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j \xi_{h,r}) \prod_{i=n+1}^p \Gamma(1 - a_i - A_i \xi_{h,r})} \times \frac{(-1)^r x^{\xi_{h,r}}}{r! B_h} \quad (2.3)$$

Where $\xi_{h,r} = \frac{b_h - r}{B_h}$

Series representation of the H-function is

$$H_{p,q}^{M,N}(z) = H_{p,q}^{M,N} \left[z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N} (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M} (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \theta(\xi) z^\xi d\xi, (z \neq 0) \quad (2.4)$$

Where $i = \sqrt{-1}$ and

$$\theta(s) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \xi) \prod_{j=1}^N \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=M+1}^Q \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=N+1}^P \Gamma(a_j - \alpha_j \xi)} \quad (2.5)$$

Here, $a_j (j = 1, \dots, P)$ and $b_j (j = 1, \dots, Q)$ are complex parameter $\alpha_j = 0 (j = 1, \dots, P), \beta_j = 0 (j = 1, \dots, Q)$ (not all zero simultaneously) and the exponents $A_j (j = 1, \dots, N)$ and $B_j (j = M + 1, \dots, Q)$ can take on integer values, reader can also refer the papers [3, 4] for more detail. The following sufficient conditions for the absolute convergence of the defining integral for the H-function given by Bushman and Srivastava [2],

$$\Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N |A_j \alpha_j| - \sum_{j=M+1}^Q |B_j \beta_j| - \sum_{j=N+1}^P \alpha_j > 0, \quad (2.6)$$

And $|\arg z| < \frac{1}{2}\Omega\pi$.

Also, from Inayat-Hussain [9], it follows that

$$H_{P,Q}^{M,N}[z] = O(|z|^{\xi*}) \text{ for small } z, \text{ where } \xi* = \min_{1 \leq j \leq M} \left[\operatorname{Re} \left(\frac{b_j}{\beta_j} \right) \right], \quad (2.7)$$

And

$$H_{P,Q}^{M,N}[z] = O(|z|^{\xi*}) \text{ for large } z, \text{ where } \xi* = \min_{1 \leq j \leq N} \left[\operatorname{Re} \left(\frac{a_j - 1}{\alpha_j} \right) \right], \quad (2.8)$$

When the exponents $A_j = B_j = 1 \forall i$ and j , the H -function reduces to the familiar Fox's H - function. Mathai *et al.* [16] introduced the following Mellin Barnes integral representation of the

Fox's **H**-function and defined by

$$\begin{aligned} H_{r,s}^{m,n}[z] &\equiv H_{r,s}^{m,n} \left[\begin{matrix} (a_r, A_r) \\ (b_s, B_s) \end{matrix} ; z \right] \equiv H_{r,s}^{m,n} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r) \\ (b_1, B_1), \dots, (b_s, B_s) \end{matrix} ; z \right] \\ &= \frac{1}{2\pi i} \int_{\Omega} \frac{\prod_{j=1}^m \Gamma(b_j + B_j \xi) \prod_{j=1}^n \Gamma(1 - a_j - A_j \xi)}{\prod_{j=m+1}^s \Gamma(1 - b_j - B_j \xi) \prod_{j=n+1}^r \Gamma(a_j + A_j \xi)} z^{-\xi} d\xi \end{aligned} \quad (2.9)$$

Where $z(\neq 0)$ and Ω is a suitable contour of integration m, n, r, s are integers such that $0 \leq m \leq s$ and $0 \leq n \leq r$. The parameters $a_1, \dots, a_r$ and $b_1, \dots, b_s$ are so constrained such that no pole of $\Gamma(b_j + B_j \xi)(j = 1, \dots, m)$ coincides with any pole of $\Gamma(1 - a_j - A_j \xi)(j = 1, \dots, n)$

Note. A detailed account of properties and applications of the H-function is available on the monograph by Mathai *et al.* [16] and Srivastava *et al.*

3. Main Result

Theorem-1 Let the condition given in (2.9) is satisfied then for $\operatorname{Re}(\gamma) > 0$ with

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$, the left sided generalized fractional derivative operator of

the Fox's H-function is given by

$$(D_{0+}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} \left\{ t^{\delta-1} H_{r,s}^{m,n} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r) \\ (b_1, B_1), \dots, (b_s, B_s) \end{matrix} ; zt \right] \right\})(x)$$

Where $z(\neq 0)$ and Ω is a suitable contour of integration m, n, r, s are integers such that

$$0 \leq m \leq s \text{ and } 0 \leq n \leq r$$

$$= x^{\delta + \alpha_1 + \alpha_2 - \gamma - 1} H_{p+3, q+3}^{m, n+3} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r), (1 - \delta, 1) \\ (b_1, B_1), \dots, (b_s, B_s), (1 - \delta + \beta_1, 1) \end{matrix} \right]$$

$$\left. \begin{matrix} (1 - \delta - \alpha_1 - \alpha_2 - \beta_2 + \gamma, 1), (1 - \delta - \alpha_1 + \beta_2, 1) \\ (1 - \delta - \alpha_1 + \beta_2, 1), (1 - \delta - \alpha_1 - \beta_2 + \gamma, 1) \end{matrix} ; zx \right] \quad (3.1)$$

Theorem-2 Let the condition given in (2.9) is satisfied then for $\text{Re}(\gamma) > 0$ with

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$, the right sided generalized fractional derivative operator

of the Fox's H-function is given by

$$(D_{0-}^{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma} \left\{ t^{-\delta} H_{r,s}^{m,n} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r) \\ (b_1, B_1), \dots, (b_s, B_s) \end{matrix} ; zt^{-1} \right] \right\})(x)$$

Where $z (\neq 0)$ and Ω is a suitable contour of integration m, n, r, s are integers such that

$$0 \leq m \leq s \text{ and } 0 \leq n \leq r$$

$$\begin{aligned} &= x^{-\delta + \alpha_1 + \alpha_2 - \gamma} H_{p+3, q+3}^{m, n+3} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r), (1 - \delta + \alpha_1 + \alpha_2 - \gamma, 1) \\ (b_1, B_1), \dots, (b_s, B_s), (1 - \delta, 1) \end{matrix} \right. \\ &\quad \left. \begin{matrix} (1 - \delta + \alpha_2 + \beta_1 + \gamma, 1), (1 - \delta - \beta_2, 1) \\ (1 - \delta + \alpha_1 + \alpha_2 - \gamma, 1), (1 - \delta - \alpha_2 - \beta_2, 1) \end{matrix} ; zx^{-1} \right] \end{aligned} \quad (3.2)$$

Proof:-(Result-I) We first replace the multivariable Fox's H-function occurring on the LHS by its Mellin-Barnes contour integrals. Changing the order of integral and derivative operator under the conditions of theorem and apply the formula for power function under generalized operator [eq. (1.3)]. The resulting Mellin-Barnes contour integrals as a Fox's H-function of multivariable We shall arrive at (3.1).

Proof:- (Result-II) Same as proof Result-I

4. Special Cases

Corollary-1. Let the conditions of Theorem 1 be satisfied and $\alpha_1 = 0, \alpha_2 = 1$ then

the Theorem 1 reduced in the following form

$$\begin{aligned} &(D_{0+}^{0,1,\beta_1,\beta_2,\gamma} \left\{ t^{\delta-1} H_{r,s}^{m,n} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r) \\ (b_1, B_1), \dots, (b_s, B_s) \end{matrix} ; zt \right] \right\})(x) \\ &= x^{\delta-\gamma} H_{p+3, q+3}^{m, n+3} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r), (1 - \delta, 1) \\ (b_1, B_1), \dots, (b_s, B_s), (1 - \delta + \beta_1, 1) \end{matrix} \right. \\ &\quad \left. \begin{matrix} (-\delta - \beta_2 + \gamma, 1), (1 - \delta + \beta_2, 1) \\ (1 - \delta + \beta_2, 1), (1 - \delta - \beta_2 + \gamma, 1) \end{matrix} ; zx \right] \end{aligned} \quad (4.1)$$

Corollary-2 Let the conditions of Theorem 2 be satisfied and $\alpha_1 = 0, \gamma = 1$ then

the Theorem 2 reduced in the following form

$$\begin{aligned}
& (D_{0-}^{0,\alpha_2,\beta_1,\beta_2,1} \left\{ t^{-\delta} H_{r,s}^{m,n} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r) \\ (b_1, B_1), \dots, (b_s, B_s) \end{matrix} ; zt^{-1} \right] \right\} (x) \\
& = x^{-\delta+\alpha_2-1} H_{p+3,q+3}^{m,n+3} \left[\begin{matrix} (a_1, A_1), \dots, (a_r, A_r), (-\delta+\alpha_2, 1) \\ (b_1, B_1), \dots, (b_s, B_s), (1-\delta, 1) \end{matrix} ; zx^{-1} \right] \\
& \quad (-\delta+\alpha_2+\beta_1+2, 1), (1-\delta-\beta_2, 1) \\
& \quad (-\delta+\alpha_2, 1), (1-\delta-\alpha_2-\beta_2, 1) ; zx^{-1} \Big] \tag{4.2}
\end{aligned}$$

REFERENCES

- [1]. **Agarwal, A. K. and Manocha, H. L.:** A new class of generating functions for generalized hypergeometric polynomials, Comment. Math. Univ. St. Paul., Vol. 28 (2)(1979) pp. 157-162.
- [2] **Buschman R.G. and Srivastava H.M.:** the H-function associated with a certain class of Feynman integrals, J.Phys. A. Math. Gen., 23(1990).
- [3]. **Erdelye, A. et al.:** Higher Transcendental Functions, Vol. I, McGraw-Hill, New York London (1953).
- [4]. **Fox C.:** The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., (1961) 98(3), 395-429 & 449-452.
- [5]. **Exton, H.:** Multiple hypergeometric functions and applications, John Wiley and Sons (Halsted Press), New York (1976).
- [6]. **Garg, M. Chanchlani, L.:** q -analogue of Saigo's fractional calculus operators, Bull. Math.
- [7]. **Garg, M. Chanchlani, L. and Alha, S.:** q -analogue of Saigo fractional integrals and derivatives of generalized basic hypergeometric series, Palestine J. Maths., Vol. 4 (1) (2015) pp. 144-155.
- [8]. **Gasper, G. and Rahman, M.:** Basic hypergeometric series, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge. (1990).
- [9]. **Inayat-Hussain A.A.:** New properties of hypergeometric series derivable from Feynman integrals: I, Transformation and reeducation formulae, J. Phys. A. Math. Gen., 20(1987), 4109-4117.
- [10]. **Kalla, S. L. Yadav, R. K. and Purohit, S. D.:** On the Reimann-Liouville fractional q -integral operator involving a basic analogue of fox H -function, Frac. Cal. App. Ana., Vol. 8 (3), (2005) pp. 313-322.
- [11]. **Khadia, S. S. and Goyal, A. N.:** On a generalized function of n variables, Vijnana Parishad Anusandhan Patrika, Vol. 13(1970) pp. 191-201.
- [12]. **Kilbas, A.A.:** Fractional calculus of generalized Wright function, Fractional Calculus and Applied Analysis, Vol. 8(2005) pp. 113-126.
- [13]. **Kiryakova, V.:** Generalized fractional calculus operators with special functions, Handbook of Fractional Calculus with Applications, <https://doi.org/10.1515/9783110571622> (2019)
- [14]. **Mathai, A. M.:** A handbook of generalized special functions for statistical and physical sciences, Clarendon Press, Oxford (1993).
- [15]. **Mathai, A. M. and Saxena, R. K.:** The H-function with application in statistics and other disciplines, John Wiley and Sons, New York (1978).
- [16]. **Mathai, A. M. Saxena, R. K. and Haubold, H. J.:** The H-function: theory and applications, Springer Science + Business Media, LLC (2010).
- [17]. **Miller, K. S and Ross, B.:** An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons inc., New York (1993).
- [18]. **Mishra, V. N. Suthar, D. L. and Purohit, S.D.:** Marichev-Saigo-Maeda fractional

calculus operators, Srivastava polynomials and generalized mittag-Leffler function,
Cogent mathematics 4(2017)

- [19]. **Pohlen, T.:** The Hadamard product and universal power series, Ph.D. Thesis, Universitat Trier, Trier, Germany (2009).
- [20]. **Rainville, E.D.:** Special functions, New York, Chelsea Publications Co. (1960).
- [21]. **Saigo, M.:** A remark on integral operators involving the Gauss hypergeometric functions. Math. Rep. College General Ed Kyushu univ., Vol.11(1978):135-143.
- [22]. **Saigo, M. and Maeda, N.:** More generalization of fractional calculus, transform Method and Special Functions., Bulgarian Academy of Sciences, Sofia Vol. 96(1998):386-400.