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DIFFERENCE CORDIAL LABELLING AND PRODUCT CORDIAL LABELLING IN SOME EXTENDED GRAPHS

Dr. M.K. Tamil Selvi*, Dr. V.M. Chitra**
*(Mathematics, Alpha College of Engineering, Chennai
Email: abcdef@gmail.com)
** (Mathematics, Velammal Vidyalaya, Chennai
Email: chithu1234@yahoo.com)

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Abstract:

This paper presents the existence of Difference Cordial Labelling and product cordial labelling, with an extensive study on labelling of Konigsberg bridge problem and Möbius–Kantor Graph. The authors investigate the difference cordial labelling behaviour of certain graphs, including path, cycle, star, and comb. The paper is based on the work of previous authors and intends to study these two properties on certain graphs.

Keywords — Graph labelling, Comb Graph, Product Cordial Labelling

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I. INTRODUCTION

Assigning integers to the vertices or edges, or both, of graphs under specific conditions is a captivating task known as graph labelling. This type of labelling plays a crucial role in mathematical modelling and finds extensive applications in areas such as cryptography, data security, astronomy, coding theory problems, communications networks, bioinformatics, and X-ray crystallography. Following the introduction of this concept, numerous journal papers have delved into diverse graph labelling techniques, including graceful labelling, magic labelling, anti-magic labelling, prime labelling, cordial labelling, odd and even graceful labelling, among others.

In 1967, Alex Rosa pioneered the exploration of Total Cordial Labelling as a strategy to address the challenge of cyclically decomposing the complete graph into trees. Over the past four decades, a substantial body of literature has emerged on this topic. Various labels have been assigned to this labelling family, including graceful, harmonious, magic, antimagic, bimagic, cordial, and prime, among others. Cordial labelling was extensively studied by Cahit in 1987, who later introduced a new label known as Total Magic Cordial Labelling in 2002.

According to Beineke and Hegde, graph labeling acts as an essential link between number theory and graph structure. Its applications go beyond mathematics, influencing various fields such as computer science and communication networks. Kalantari, Khosrovshahi, and Mitchell sought to apply magic labelling in optimization theory, particularly in addressing the traveling salesman problem. Baskoro et al. proposed a construction for a secret sharing scheme using edge magic labelling. More recently, Hartnell and Rall introduced a game based on vertex magic labelling.

In this paper, we present the demonstration of the existence of Difference Cordial Labelling and product cordial labelling. An extensive study was done on labelling of Konigsberg bridge problem and Möbius–Kantor Graph. This is the key point to write this paper which the part of the work has already been done by many recent authors. We intend to study these two properties on certain graphs.

II. PRELIMINARIES

Throughout this paper the letters v_i, e_j denote the name of the vertices and edges respectively where i and j are natural numbers.

Definition 1: The ladder graph L_m is a planar undirected graph (P_m) with 2m vertices and 3m – 2 edges. It is obtained as the Cartesian product of two path graphs, $L_m=P_m \times P_1$, one of which has only one edge: where m is the number of rungs in the ladder.

Definition 2 (Duplicate graph) Let G (V, E) be a simple graph and the duplicate graph of G is DG = (V_1, E_1),where the vertex set $V1 = V \cup V'$ and $V \cap V' = \phi$ and $f : V \to V'$ is bijective (for $v \in V$, we write f (v) = v′ for convenience) and the edge set E_1 of DG is defined as the edge *ab* is in E if and only if both *ab′* and *a′b* are edges in E_1.

Definition 3: Comb graph is a graph obtained by joining a single pendant edge to each vertex of a path.

Definition 4: Let EDG = (V_1, E_1) be a duplicate graph of the comb graph G (V, E). We add an edge between any one vertex from V to any other vertex in V' except the terminal vertices of V and V'. For convenience, we take $v_i \in V$ and $v_j' \in V'$ and thus the edge $v_i v_i'$ is formed. We call this new graph as the extended duplicate graph of the Comb C_m and it is denoted by EDG (C_m). Clearly | Vm| = 2m and |Em| = 2m-1, where 'm' is the number of internal vertices of comb.

Definition 5: A binary vertex labelling of a graph G is called a cordial labelling if$| v f(0) – v f (1)| \leq 1$ and $|e f(0) – e f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling

Definition 6: Difference cordial labelling: Let be a graph. Let $f : V(G) \to \{1,2,3,\dots p\}$ be a function. For each (u,v) assign the label |f(u)-f(v)| , f is called difference cordial labelling if f is one to one and the absolute difference of number of edges labelled with1 and edges not labelled with 1 is at most 1. Any graph with difference cordial labelling is called difference cordial graph.

Definition 7: Let G = (V(G),E(G)) be a graph with p vertices. A bijection $f :V(G) \to \{1,2,\dots p\}$is called a prime labelling if for each edge e = uv, gcd{f (u), f (v)}=1. A graph which admits prime labelling is called a prime graph.

Definition 8: Triangular snake graph T_n with pendent edges is formed by vertex set $V(T_n) = \{u_i , v_i ; 1 \leq i \leq n-1\} \cup \{s_i , t_i ; 1 \leq i \leq n\}$ and edge set $E(Tn) = \{u_i v_i ; 1 \leq i \leq n-1\} \cup \{s_i t_i ; 1 \leq i \leq n-1\} \cup \{v_i s_{i+1} ; 1 \leq i \leq n-1\} \cup \{s_i s_{i+1} ; 1 \leq i \leq n-1\} \cup \{s_i t_i ; 1 \leq i \leq n\}$. It has $(5n-4)$ edges and $(4n-2)$ vertices.

Definition 9: A quadrilateral snake Q_n is obtained from a path $a_1, a_2, \dots a_n$ by joining, a_i, a_{i+1} to new vertices b_i and c_i respectively and adding edges $b_i c_i$ for i = 1, 2, …, n – 1. That is every edge of a path is replaced by a cycle.

III. MAIN RESUKTS

Konigsberg Bridge Problem (KBP):

Two islands linked to each other with seven bridges. The problem is to start from any one of the four land areas, take a stroll across the seven bridges and get back to the starting point without crossing a bridge second time. This is called Konigsberg Bridge Problem and we abbreviate the problem as KBP.[3] We represent the vertices and edges of the Konigsberg bridge problem as V = {v1,v2,v3,v4} ; E = {e1,e2,e3,e4,e5,e6,e7}

The KBP has Prime Labeling.

Let us label the graph G as follows $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$ which is shown in Fig. 1. This fact is already proved in [5].

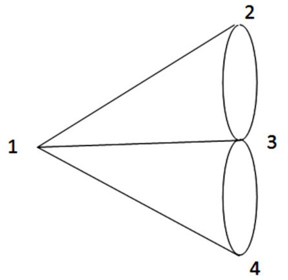

Fig.1 KBP Diagram

Theorem 1: The KBP with Prime labelling is difference cordial labelling

When the edges are assigned the labels as per the definition we see that f is one to one and the number of edges labelled with 1 is 3 and the number of edges not labelled with 1 is 2 Note that we ignore the parallel edges. Hence KBP with prime labelling is difference cordial labelling.

Theorem 2: The following Fig. 2 show the triangular Ladder graph (TL_6) and its extended duplicate graph EDG(TL_m).

Proof: Clearly the duplicate graph of the triangular ladder graph TL3 contains 2m=6 vertices and 4m – 3=9edges.

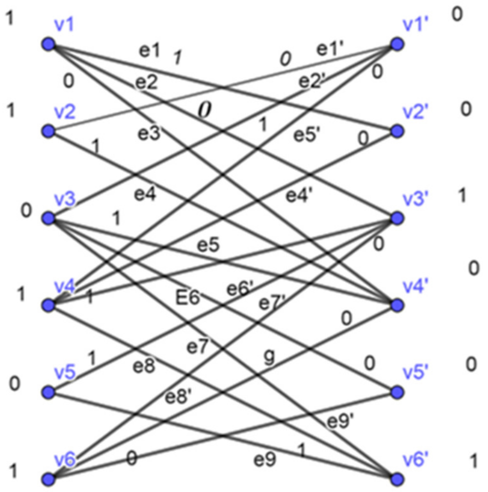

Fig.2 EDG(TL_m) Diagram

The edges are labelled as |f(u)-f(v)|. Thus the number of edges labelled 1 and 0 are equal. Hence the duplicate graph is difference cordial graph.

Theorem 3: Triangular snake graph $\boldsymbol{Tn}$, > 1 with pendant edges is cordial graph which was proved in [7]. This graph is also a difference cordial graph.

Proof: Let Tn be a triangular snake graph with pendent edges. Now we label all edges as

$$f(s_i s_{i+1}) = \begin{cases} 1, if \quad i = \dfrac{n-1}{2} and\ i = \dfrac{n+1}{2}, \\ 0, \qquad\qquad\qquad otherwise \end{cases}$$

$$f(s_i t_i) = 1, \forall\, i$$

$$f(u_i v_i) = 1, \forall\, i$$

$$f(u_i s_i) = 0, if\ 1 \le i \le n$$

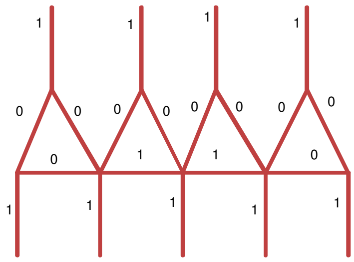

Fig.3 Triangular Snake Graph

From the Fig. 3 it is clear that the graph is difference cordial graph since f is one to one and all the edges are labelled as |f(u)-f(v)|. And number of edges labelled with 0 is 2n+1 and number of edges labelled with 1 are 2n and the difference is 1

This graph is not product cordial labelling since the edge labelling is done such that the difference between the number o edges labelled with 0 and 1 is at most 1. So product cordial labelling is not possible.

Theorem 4: Quadrilateral snake graph Q_n is a difference cordial graph. Refer to Fig. 4.

Proof: Now we label all vertices as

$$f(ai) = \begin{cases} 0, & if i = 1,2 \\ 1, if i \equiv 3(mod4), & i \equiv 6(mod4) \\ 0, & otherwise \end{cases}$$

$$f(bi) = \begin{cases} 1, & if i = 1 \\ 1, if\ 2 \leq i \leq n, even \\ 0, if\ 3 \leq i \leq n, odd \end{cases}$$

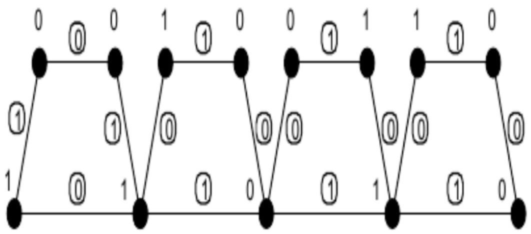

Fig.4 Quadrilateral Snake Graph

Theorem 5: Quadrilateral snake graph is not product cordial

Proof*:* Each quadrilateral has labelling as equal number of 1 and 0. This implies that except 1 edge other edges will have 0 labelling according to product label definition. So it is not product labelling.

Theorem 6 : The n-sunlet graph is the graph on n vertices obtained by attaching n pendant edges to a cycle graph $\mathbf{C_n}$

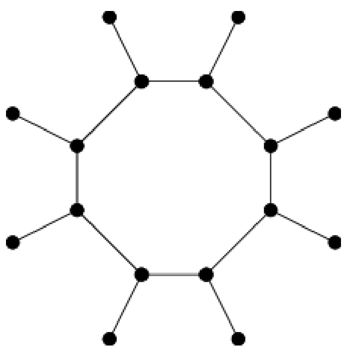

Fig.5 n-Sunlet Graph

Proof*:* The vertices in the inner cycle are labelled as 1, 2, 3, 4, 5 …n. The pendant edge connected to the vertex labelled as 1 to be labelled as n+1 and sequential labelling to be done. So the all the edges of the cycle will be labelled s 1 and all the pendent edges will be labelled as n. So the n-sunlet graph is difference cordial graph.

IV. MÖBIUS–KANTOR GRAPH

The Möbius–Kantor graph is a symmetric bipartite cubic graph with 16 vertices and 24 edges. It can be defined as a subgaph of the four-dimensional hypercube graph, formed by removing eight edges from the hypercube. The graph is also the bipartite point-line incidence graph of the Möbius–Kantor configuration, which is a configuration consisting of eight points and eight lines, with three points on each line and three lines through each point. The Möbius–Kantor graph is the unique cubic symmetric graph on 16 nodes and has various properties such as being Hamiltonian, having a chromatic number of 2, and a chromatic index of 3

Theorem 7: The Möbius–Kantor graph is difference cordial labelling in Fig.5

Proof: The labelling is done in the following way

- The vertices v_i on the outer circle are labelled as 1 if i is 1(mod 2).
- They are labelled as 0 if i is 0 mod (2).

- Now the vertices v_j in the inner circle are labelled as 0 if j is 1 mod (2) and 1 if j is 0 mod(2).
- The edges $(v_i, v_j) = 1$ if vi and vj have the same label and 0 otherwise on the outer circle.
- In the inner circle there are eight edges.
- These edges (v_i, v_j) are labelled as 1 if i or j or both is 0 mod(3).
- Otherwise the edges are labelled as 0.

With these labelling the Möbius–Kantor graph is difference cordial labelling is proved.

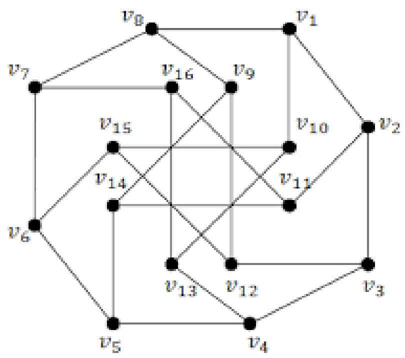

Fig.5 Möbius–Kantor graphis

VII. CONCLUSION

This research contributes to the broader understanding of graph labelling, particularly difference cordial labelling. By proving these theorems, we have established a framework that can be applied to other complex graph structures, potentially aiding in solving more intricate problems in graph theory. Future work may focus on exploring other labelling techniques and their implications on different graph properties, further enriching the **field** of mathematical graph theory.

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