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A Brief Review of Differentiable Manifolds and Their Applications

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Abstract

The main idea of this review paper is to make understandable the idea of differentiable manifolds using Lie groups . As recounted in this paper, the idea of groups is one that has evolved from some very intuitive concepts. We can do binary operations like adding or multiplying two elements and also binary operations like taking the square root of an element (in this case the result is not always in the set). In this paper, we aim to find the operations and actions of Lie groups on manifolds. These actions can be applied to the matrix group and Bi-invariant forms of Lie groups and to generalize the eigenvalues and eigen functions of differential operators on n . A Lie group is a group as well as differ-entiable manifold, with the property that the group operations are compatible with the smooth structure on which group manipulations, product and inverse, are distinct.This paper presents an overview of differentiable manifolds, focusing on their definition, prop-erties, and significance. Beginning with the basic definitions of smoothness and charts, we explore the construction of differentiable structures on topological spaces. We delve into tangent spaces, differential forms, and vector fields, elucidating their role in defining smooth functions and mappings between manifolds. Additionally, we discuss important theorems such as the inverse function theorem and Stokes' theorem, showcasing their relevance in manifold theory. Finally, we touch upon applications in geometric model-ing, general relativity, and gauge theory, underscoring the manifold's importance across disciplines. It plays an extremely important role in the theory of fiber bundles and also finds vast applications in physics. It represents the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical phys-ics. Here we did work flat out to represent the mathematical aspects of Lie groups on manifolds.Differentiable manifolds, a fundamental concept in differential geometry, find widespread applications across various fields of mathematics, physics, and engineering. This abstract provides a concise overview of some notable applications of differentiable manifolds.

As in theoretical physics, differentiable manifolds serve as the mathematical frame-work for formulating theories such as general relativity. The spacetime continuum is modeled as a four-dimensional differentiable manifold, where curvature encodes gravita-tional effects. This application has profound implications in understanding the dynamics of celestial bodies, black holes, and the structure of the universe. In robotics and control theory ifferentiable manifolds play a crucial role in modeling the configuration spaces of robotic systems. By representing the possible configurations of robot joints as points on a manifold, researchers can develop efficient algorithms for motion planning, trajectory optimization, and control synthesis. These applications find practical use in robotic ma-nipulation, autonomous vehicles, and industrial automation. In machine learning and data analysis manifold learning techniques aim to uncover the underlying structure of high-dimensional data by approximating it as a lower-dimensional manifold embedded in the ambient space.

Algorithms such as t-SNE (t-distributed Stochastic Neighbor Em-bedding) and Isomap leverage differential geometry principles to perform dimensionality reduction and visualize complex datasets. These methods facilitate exploratory data ana-lysis, pattern recognition, and feature extraction tasks in machine learning.

Keywords Group, Abelian Group,Lie Groups, Smooth Mapping, tangent space, directional derivat-ive.

Introduction

In the present era, the study of the group related to the Lie group is essential for the sake of its comprehensive applications in several fields.

Through the mathematical analysis, representations of groups on the manifold are vital due to it allows many group-theoretic problems in form of linear algebra problems, which is well perceived. However, Lie groups were studied by Marius Sophus Lie (1842-1899) for the very first time , who used it to solve ordinary differential equations. Lie groups and Lie algebras, together with acquainted Lie theory which plays an effective role in the branch of pure and applied mathematics that is utilized in modern physics as well as an active area of research. This is an introductory course on differentiable manifolds.By the literature view, we can observe that many researchers have endeavored to analyze and discuss the importance of operations and actions of Lie groups on manifolds in various fields.In the present tasks, we discussed the Group and Abelian groups also Subgroups with the Lie

groups based on the smooth manifolds that assist us to understand the properties and actions of the Lie groups on Manifolds. We also reveal some example which shows a clear view of operations and actions of Lie groups on Manifolds.

This paper contains the following sections. In the first section I will try to explain the inside details of algebraic structure of g. In the next section I will discuss their advantages and disadvantages. At the end, Conclusion will be given.

1. Group

Group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces.

A group is a set G equipped with a binary operation: $G \times G \to G$ that associates an element ab $\in G$ to every pair of elements $a, b \in G$, and having the following properties:
$\cdot$ is associative, has an identity element $e \in G$, and every element in G is invertible (with respect to $\cdot$). More explicitly, this means that the following equations hold for all
$a, b, c \in G$:

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity).
(ii) $ae = ea = a$ (identity). (1)

(iii)F or every $a \in G$, there is some $a^{-1} \in G$ such that $^{-1} = a^{-1}a = e$. (inverse). (2)

Otherwise, we say two elements, g_1 and g_2 of a group commute with each other if their product is independent of the order, i.e., if $g_1g_2 = g_2g_1$. If all elements of a given group commute with one another then we say that this group is Abelian.

1.0.1 Example

The real numbers under addition or multiplication (without zero) form an Abelian group. The cyclic groups Zn are Abelian for any n (n is a natural number). The symmetric group Sn is not Abelian for n > 2 , but it is Abelian for n = 2. Simplexes are building blocks of a polyhedron.

1.1 Isomorphism and Homomorphism in Groups

1.1.1 Isomorphism

Two groups G1 and G2 are isomorphic if we can put their elements into a one-to-one cor-respondence which is preserved under the composition laws of the groups. The mapping

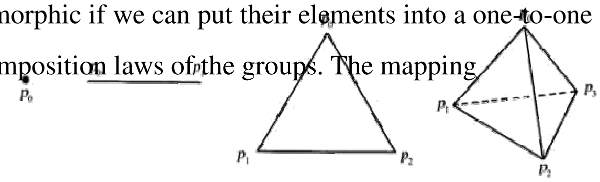

Figure 1: The elements of S2

between these two groups is called an isomorphism.

1.1.2 Homomorphism

If group G1 being mapped into another group G2 but not in a one-to-one manner, i.e. two or more elements of G1 are mapped into just one element of G2. If such mapping respects the product law of the groups we say they are homomorphic. The mapping is then called a homomorphism between G and G'.

2. Subgroup

Given a group G, a subset H of G is a subgroup of G if and only if:

1. The identity element e of G also belongs to H ($e \in H$);
2. For all $h_1, h_2 \in H$, we have $h_1h_2 \in H$;
3. For all $h \in H$, we have $h^{-1} \in H$.

It is easily checked that a subset $H \subseteq G$ is a subgroup of G if and only if H is nonempty and whenever $h_1, h_2 \in H$, then $h_1h_2^{-1} \in H$.

2.1 Definition

A subset H of a group G which satisfies the group postulates under the same composition law used for G, is said to be a subgroup of G. The identity element and the whole group G itself are subgroups of G. They are called improper subgroups. All other subgroups of a group G are called proper subgroups. If H is a subgroup of G, and K a subgroup of H, then K is a subgroup of G. article

2.2 Theorem:

The order of a subgroup of a finite group is a divisor of the order of the group.

Proof:

For a finite group G of order m with a proper subgroup H of order n, we can write m = kn, where k is the number of disjoint sets gH. The set of elements gH are called left cosets of H in G. They are certainly not subgroups of G since they do not contain the identity element, except for the set eH = H. Analogously, we could have split G into sets Hg, which are formed by elements of G that differ by an element of H multiplied from the left. The same results would be true for these sets. They are called right cosets of H in G. The set of left cosets of H in G is denoted by G/H and is called the left coset space. An element of G/His a set of elements of G, namely gH.

Analogously, the set of right cosets of H in G is denoted by G/H and it is called the right coset space. If the subgroup H of G is an invariant subgroup, then the left and right cosets are the same since $g1Hg^{-1}H = e$ implies gH = Hg. In addition, the coset space G/H, for the case in which H is invariant, has the structure of a group and it is called the factor group or the quotient group. To show this, we consider the product of two elements of two different cosets. We get

$$gg'Hg''H \equiv (gg'H)g''H = g(g'Hg''H). \quad (3)$$

where we have used the fact that H is invariant, and therefore there exists $h \in H$ such that $gg' = g''h$. Thus, we have obtained an element of a third coset, namely $gg''H$. If we had taken any other elements of the cosets gH and $g'H$, their product would produce an element of the same coset $gg''H$. Consequently, we can introduce, in a well-defined way, the product of elements of the coset space G/, namely

$$(gH)(g'H) \equiv gg''H. \quad (4)$$

The invariant subgroup H plays the role of the identity element since

$$(gH)(H) = gH. \quad (5)$$

The inverse element is $g_{-1}H$ since

$$(gg^{-1}H)(H) = gH. \quad (6)$$

The associativity is guaranteed by the associativity of the composition law of the group G. Therefore, the coset space G/H and G/H is a group in the case where H is an invariant subgroup. Such a group is not necessarily a subgroup of G or H.

This completes the proof of the theorem.

3. Lie Group

Let G be a differentiable manifold having group structre on it that the group operations.

i) $G\times G \rightarrow G$, $(g1,g2) \rightarrow g1.g2$ i

i) $G \rightarrow G$, $g \rightarrow g_{-1}$

The elements of Lie group G are differentiable. It can be shown that G has a unique analytic structure with which the product and the inverse operations are written as a convergent power series.

The unit element of a Lie group is written as e. The dimension of a Lie Group G is defined to be the dimension of G as a manifold. The product symbol may be omitted, and g_1g_2 is usually written as g_1g_2.

3.1 Example of Lie groups

1. Rn with addition as the group operation and with the usual differential structure is a Lie group.
2. S1 considered as the subset of

   z € C :|z| = 1 of C with the usual multiplication in C as the group operation and with the smooth structure introduced is a Lie group.
3. A most important example is G7(n,IR). This is a Lie group with the smooth struc-ture and with the matrix multiplication as its group operation. You should check the détruis of this example instead of taking for granted, as this will make you realize the significance of some of our earlier exercises and examples.
4. The manifold O(n,IR) is a Lie group with the smooth structure we have introduced earlier. In a similar way we can turn U(n), SU(n) and Sl(n,IR) into smooth man-ifolds. They are all Lie groups with these smooth structures and with the matrix multiplication. The reader should try to prove this. This is a special case of Cartan's theorem which says that any closed subgroup of a Lie group is a Lie (sub)group.

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3.2 Theorem

SL(n, R) is a Lie group.

Proof

To prove that SL(n, R) is a Lie group, we need to show that it is a smooth manifold and that the group operations are smooth.Let f : $GL_n \rightarrow R$ be given by f(A) = det(A). The level set $f^{-1}(1)$ is given by,

$$\{A \in SL_n \mid \det(A) = 1\}.$$

The Special Linear Group. The derivative of f is surjective at a point $A \in GL_n$, making SL(n) into a Lie group. Such that

$$\lim_{h\to 0} \frac{\det(I + hB) - \det(I)}{h} = \mathrm{tr}(B),$$

implying that

$$\lim_{h\to 0} \frac{\det(A + hB) - \det(A)}{h} = \lim_{h\to 0} \frac{\det(1 + hA^{-1}B) - 1}{h} = \mathrm{tr}(A^{-1}B),$$

since det(1) = 1 for any $k \in R$. We can take the matrix kI − A to obtain

$$A \cdot df(B) = \mathrm{tr}(I) \cdot (kI - A)^{-1},$$

therefore df(A) is surjective for every $A \in SL_n$. Consequently, SL(n) is a submanifold of GL(n). Therefore, the group multiplication and inversion are differentiable, so SL(n) is a Lie Group.

4. Tangent Space

The tangent space T_pM at a point p in a manifold M is defined as the set of all tangent vectors to M at p. article amsmath

The tangent space at a point p is denoted by T_p.article amsmath amssymb

Let M be a smooth manifold, and $p \in M$. Let $C^\infty(p)$ be the set of R-valued smooth functions defined in a neighborhood of p. Thus, if $f \in C^\infty(p)$, then there exists a neighborhood U_f (which depends on f) of p such that f is defined and C^∞ on U_f .

4.1 Definition

A tangent vector v at a point $p \in M$ is a mapping from $C^\infty(p)$ to R enjoying the following properties:

1. $v(f) \in R$, for all $f \in C^\infty(p)$.
2. $v(af + bg) = av(f) + bv(g)$ for $a, b \in R$ and $f, g \in C^\infty(p)$. This equality is on R.
3. $v(f\,g) = f(p)v(g) + v(f)g(p)$.

Thus a tangent vector is a "linear functional" on $C_\infty(p)$ (by 1) and 2)) and it satisfies a Leibnitz type rule 3). Now the question is whether there exists any nonzero tangent vector at p. (0 defined by 0(f) = 0 is trivially a tangent vector!) Let us denote by T_pM the set of all tangent vectors at p to M. We wish to show that T_pM has nontrivial elements.

Let (U, φ) be a coordinate chart. We say it is centered at p if φ(p) = 0. (There always exist such charts.) Let x_i be the corresponding coordinate functions on U. Then for any

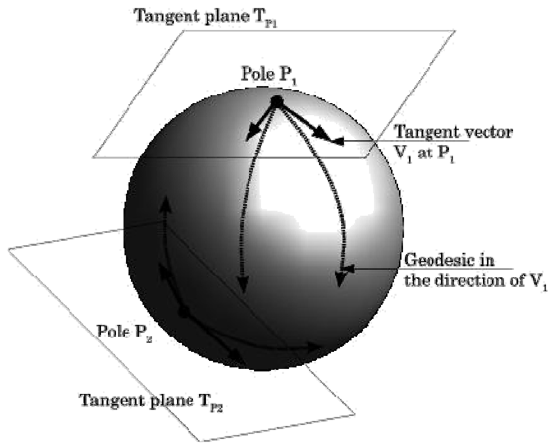

Figure 2: Illustration of tangent space ,tangent vector and geodesics.

$f \in C^\infty(p)$, we define

$$v(f) := \frac{\partial f}{\partial x_i}(p).$$

Then it is easy to check that $v \in T_pM$. They are nontrivial since if we take $f = x_i$, then $v(x^i)$ –

$$v(x^j) = \frac{\partial x_i}{\partial x^k}(0) = \delta^i_k .$$

More generally, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be any smooth curve through p. Then we define $\gamma'(0)$ by setting for any $f \in C^\infty(p)$

$$(\gamma'(0))(f) := \frac{d}{dt}\Big|_{t=0} (f \circ \gamma).$$

We claim that γ′(0) is a tangent vector at p.

$$\begin{aligned} \gamma'(0)(f + tg) &= \gamma'(0)(f) + \gamma'(0)(tg) \\ &= (\gamma'(0))(f) + (\gamma'(0))(g) \\ &= \frac{d}{dt}\Big|_{t=0}(f \circ \gamma) + \frac{d}{dt}\Big|_{t=0} (g \circ \gamma) \\ &= \frac{d}{dt}\Big|_{t=0}((f + tg) \circ \gamma) \\ &= \gamma'(0)(f) + \gamma'(0)(g). \end{aligned}$$

Now why is this more general than the earlier example? For, the tangent vector $v_i|_p$ is $\gamma'(0)$ of the i-th coordinate curve γ : t (0, ..., 0, t, 0, ..., 0). (Exerc

It is very easy to see that the set T_pM of tangent vectors at p to M form a vector space over R in an obvious way. For $a \in R$ and $v_1, v_2 \in T_pM$, we set $av(f) := a \cdot v(f)$ and $(v\gamma'(\theta)$ and $(v_1 + v_2)(f) := v_1(f) + v_2(f)$.

4.2 Example

. Consider the sphere

$$S^2 = x^2 + y^2 + z^2 = 1$$

in R^3.

We want to find the plane passing through the north pole N(0,0,1) that is "closest" to the sphere.

The classics would refer to such a plane as an osculator plane.

The natural candidate for this osculator plane would be a plane given by a linear equation that best approximates the defining equation

$$x^2 + y^2 + z^2 = 1$$

in a neighborhood of the North pole. The linear approximation of

$$x^2 + y^2 + z^2$$

near N seems like the best candidate. We have

$$x^2 + y^2 + z^2 - 1 = 2(z - 1) + O(2)$$

Hence, the osculator plane is z = 1,

Geometrically, it is the horizontal affine plane through the North pole. The linear subspacez $= 0 \subset R_3$ is called the tangent space to S_2atN.

Conclusion

Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. From this paper, we have seen that Lie group embodies three different forms of mathematical structure. Firstly, it has a group structure. Secondly, the elements of this group also form a "topological space" so that it may be described as being a special case of "topological group". Finally, the elements also constitute an "analytic manifold". More generally Lie groups are differentiable manifolds which have very important consequences on Manifolds which are locally Euclidean spaces. We have summarized that by using differentiable structure, we can approximate the neighbourhood of any point of a Lie group by a Euclidean space which is the tangent space to the Lie group at that particular point. This approximation is some sort of local linearization of the Lie group.Thus we see that Lie groups provides us tools to generalised the abstract concepts in mathematical way.

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