compatible mappings on a 2-Metric Space

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Abstract: The aim of the present paper is to obtain i) a common fixed point theorem for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem using the concept of joint reciprocal continuity in 2- metric spaces. A supporting example is also given.

Keywords: common fixed point , Compatible mappings and asymptotic regularity.

Mathematical subject Classification : 47 H 10, 54H25

1. Introduction : Gähler[2] introduced the concept of 2-metric space as a natural generalization of a metric space. Some fixed point theorems in 2-metric spaces are obtained in Iseki[3],Rhoades[5] and Jungck[4]. Gähler[2] introduced the notions of reciprocal continuity and asymptotic regularity for a pair of self maps on a 2-metric space. Using the above concepts Badshah and Gopal Meena[1] proved a fixed point theorem for a pair of self-maps on a 2-metric space using asymptotic regularity.

The present paper is a generalization of a result of Badshah and Gopal Meena([1],Theorem 2).

In this paper, we obtain i) a common fixed point theorem (Th. (2.1)) for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem (Th. (2.6)) using the concept of joint reciprocal continuity. We start with some definitions.

Definition 1.1 (Gähler[2]): Let X be a non-empty set with real valued function $d: X^3 \to R$ satisfying :

i) For two distinct points $x, y\ in\ X,\ \ there\ exists\ z\ in\ X\ such\ that\ d(x, y, z) \neq 0$
ii) $d(x, y, z) = 0$ only if at least two of $x, y\ and\ z\ are\ equal$
iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ and
iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u\ in\ X.$

The function d is called a 2-metric on X and the pair (X, d) is called a 2-metric space.

Definition 1.2 (G$\ddot{a}$hler[2]) : Let (X, d) be a 2-metric space .

i) A sequence $\{x_n\}$ is said to be convergent to a point x in X if $= 0, \forall a \in X$.

ii) A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if $d(x_m, x_n, a) = 0, \forall a \in X$.

iii) A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Note: i) In a 2-metric space (X, d), $d: X^3 \rightarrow R$ is continuous if

$$x_n \rightarrow x,\ y_n \rightarrow y \text{ implies } d(x_n, y_n, a) \rightarrow d(x, y, a), \forall a \in X \text{ for } n \rightarrow \infty.$$

ii) If $x_n \rightarrow x$ and $x_n \rightarrow y$ as $n \rightarrow \infty$ then $x = y$.

Definition 1.3 (G$\ddot{a}$hler[2]): For self-mappings, S and T of a 2-metric space (X, d), the pair (S, T) is called reciprocally continuous if
$d(STx_n, Sx, a) = 0 = \forall a \in X$, whenever $\{x_n\}$ is a sequence in
X such that $S x_n = T x_n = x$, for some $x \in X$.

Definition 1.4 (G$\ddot{a}$hler[2]): For self-mappings , S and T of a 2-metric space (X, d)

i) A sequence $\{x_n\}$ in X is called asymptotically regular with respect to the pair (S, T) if $d(Sx_n, Tx_n, a) = 0\ \forall a \in X$.

ii) The pair (S, T) is called compatible if $d(STx_n, TSx_n, a) = 0\ \forall a \in X$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n = Tx_n = x$, for some $x \in X$.

Definition 1.5: Suppose P, S and T are self maps on a 2-metric space (X, d).The pair (S, T) is said to be joint reciprocal continuous with respect to P if there exists a sequence $\{x_n\}$ in X such that $Sx_n = Tx_n$ and $d(PSx_n, SPx_n, a) = 0$ and $d(PTx_n, TPx_n, a) = 0\ \forall a \in X$.

Note : If $\{x_n\}$ is a sequence in X such that $Sx_n = Tx_n$

and (P, S) and (P, T) are compatible then (S, T) is jointly reciprocally continuous with respect to P.

Notation: $\Phi = \{\varphi: [0, \infty) \rightarrow [0, \infty),\ \phi \text{ is continous and } \phi(t) < t\ \forall t > 0\}$.

2. Main Results

We begin with our first main result, from this we obtain corollaries of the main result and finally show that the result of Badshah and Gopal Meena[1] follows as a corollary.

Theorem 2.1: Let P, S and T be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)$ $d(Px, Py, a) \leq$ φ ({ $d(Py, Ty, a)(1 + d(Px, Sx, a))$, $d(Px, Sx, a)(1 +$
)

$\forall x, y, a \in X$ and for some $\varphi \in \Phi$,

$ii)$ the pair (P, S) and (P, T) are compatible

$iii)$ there exists a sequence $\{x_n\}$ which is asymptotically regular with respect (P, S) and (P, T)

$iv)$ S and T are continous $v)$ d is continous.

Then P, S and T have a unique common fixed point in X.

Proof: Let $\{x_n\}$ be a sequence in X satisfying condition (iii).

By taking $x = x_n$ and $y = x_m$ in (i), we obtain

$$d(Px_n, Px_m, a) \leq \varphi\{ d\left(Px_m, Tx_m, a\right)\left(1 + d\left(Px_n, Sx_n, a\right)\right),\ d\left(Px_n, Sx_n, a\right)\left(1 + \qquad d\left(Px_m, Tx_m, a\right)\right) \quad (2.1.1)$$

On letting $n \to \infty$ and using condition (iii), we get

$$d(Px_n, Px_m, a) = 0 \ \forall a \in X.$$

This implies that the sequence $\{Px_n\}$ is a Cauchy sequence in X, since X is complete with respect to 2-metric d.

This implies

$$Px_n \to z, \ \ for\ some\ z \in X. \qquad (2.1.2)$$

Now

$$d\left(Sx_n, z, a\right) \leq d\left(Sx_n, z, Px_n\right) + d\left(Sx_n, Px_n, a\right) + d\left(Px_n, z, a\right)$$

On letting $n \to \infty$, using condition (iii) and equation (2.1.2), we get

$$d(Sx_n, z, a) = 0 \ \forall a \in X.$$

This implies $Sx_n \to z$ in X. (2.1.3)

Similarly we can obtain

$Tx_n \to z \; in \; X.$ (2.1.4)

Since we can write

$$d(PSx_n, Sz, a) \leq \; d(PSx_n, Sz, SPx_n) + \; d(PSx_n, SPx_n, a) + \; d(SPx_n, Sz, a)$$

On letting $n \to \infty$, using the conditions (ii), $(iv) and \; (v)$, we get

$d(PSx_n, Sz, a) \; = 0 \; \forall a \in X,$

$(\because Px_n \; \to z \; and \; S \; is \; continuous \; implies \; SPx_n \; \to Sz)$

This implies $PSx_n \to Sz$. (2.1.5)

Similarly we can prove $PTx_n \to Tz.$ (2.1.6)

By taking $x \; = \; Sx_n \; and \; y \; = \; Tx_n$ in (i), we obtain

$$d(PSx_n, PTx_n, a) \; \leq \; \varphi(max\{d(PTx_n, TTx_n, a)(1 \; + d(PSx_n, SSx_n, a)),$$

$$d(PSx_n, SSx_n, a)(1 \; + d(PTx_n, TTx_n, a))\}) \qquad (2.1.7)$$

From condition (iv) we have $S \; and \; T$ are continuous, applying continuity of $S \; and \; T$ in (2.1.3) and (2.1.4) ,we get

$SSx_n \; \to Sz, \; TTx_n \; \to Tz \; and \; STx_n \; \to Sz$ (2.1.8)

On letting $n \to \infty$ in (2.1.7), using (2.1.5),(2.1.6),(2.1.8) and condition (v), we get

$$d(Tz, Sz, a) = \; 0, \forall a \in X \; .$$

$$This \; implies \; Tz \; = \; Sz. \qquad (2.1.9)$$

Similarly by taking $x \; = \; Tx_n \; and \; y \; = \; z \; in \; (i)$, we get

$$d(PTx_n, Pz, a) \leq \; \varphi(\{d(Pz, Tz, a)(1 \; + \; d(PTx_n, STx_n, a), d(PTx_n, STx_n, a)(1 \; + \; d(Pz, Tz, a)\})$$

On letting $n \to \infty$, using ((2.1.7),(2.1.8),(2.1.9) and condition(v), we get
$d(Tz, Pz, a) = \; 0, \forall a \in X$.

$This \; implies \; Tz \; = \; Pz.$ (2.1.10)

Therefore $Sz \; = \; Tz \; = \; Pz.$ (2.1.11)

By taking $x \; = x_n \; and \; y \; = \; z \; in \; (i)$, we get

$$d(Px_n, Pz, a) \leq \; \{d(Pz, Tz, a)(1 \; + \; d(Px_n, Sx_n, a), \; \; d(Px_n, Sx_n, a)(1 \; + \; d(Pz, Tz, a)\})$$

On letting $n\to\infty$ and using (2.1.2),(2.1.3),(2.1.11) and condition(iv), we get

$d(z, Pz, a) = 0, \forall a \in X$.$This\ implies\ \ Pz = z.$

Therefore $z\ is\ a\ fixed\ point\ of\ P.$ Hence z is a common fixed point of $S, T\ and\ P\ in\ X.$

Suppose if x is common fixed point of $S, T\ and\ P\ in\ X.$ Then it can be easily proved that $x = z$. Hence z is a unique common fixed point of $S, T\ and\ P\ in\ X.$

Now we have the following corollaries of theorem 2.1.

Corollary 2.2: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\ d(Px, Py, a) \leq\ \varphi(\{\ d(Py, Ty, a)(1 + d(Px, Sx, a)),\ \ d(Px, Sx, a)(1 + d(Py, Ty, a), d(Px, Sx, a), d(Py, Sy$

$\forall x, y, a \in X\ and\ \ for\ some\ \varphi \in \Phi,$ and also satisfying condition $(ii), (iii), (iv) and\ (v)$ of theorem 2.1.

Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.3 : Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\ d(Px, Py, a) \leq\ \lambda\{\ d(Py, Ty, a)(1 + d(Px, Sx, a)),\ \ d(Px, Sx, a)(1 + d(Py, Ty, a), d(Px, Sx, a), d(Py, Sy,$

$\forall x, y, a \in X$, $0 < \lambda < 1$ and also satisfying condition $(ii), (iii), (iv) and\ (v)$ of theorem 2.1.

Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.4: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\ d(Px, Py, a) \leq\ \alpha\{\ d(Py, Ty, a)(1 + d(Px, Sx, a)),\ \ d(Px, Sx, a)(1 + d(Py, Ty, a)\}\ \ + \beta\, max\{d(Px, Sx,$

$\forall x, y, a \in X$, $\alpha\ and\ \beta\ are\ non-negative\ numbers\ such\ that\ \alpha + \beta < 2 and$

also satisfying condition $(ii), (iii), (iv) and\ (v)$ of theorem 2.1.

Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.5: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\ d(Px, Py, a) \leq\ \alpha\beta\{d(Px, Sx, a) + d(Py, Sy, a)\}$

$\forall x, y, a \in X$, $\alpha\ and\ \beta\ are\ non-negative\ numbers\ such\ that\ \alpha + \beta < 2$ and also satisfying condition $(ii), (iii), (iv) and\ (v)$ of theorem 2.1.

Then $P, S\ and\ T$ have a unique common fixed point in X.

The following result due to Badshah and Gopal Meena[1] is a corollary of the above result.

Corollary 2.6 (Badshah and Gopal Meena[1],Theorem 2): Let P, S and T be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)$ $d(Px, Py, a) \leq \alpha \frac{d(Py,Ty,a)(1+d(Px,Sx,a))}{1+d(Sx,Ty,a)}$

$$+ \beta\{d(Px, Sx, a) + d(Py, Sy, a\}$$

$\forall x, y, a \in X$, α and β are $non-negative$ $numbers$ $such$ $that$ $\alpha + \beta < 2$,

and also satisfying condition (ii), (iii), (iv) and (v) of theorem 2.1.

Then P, S and T have a unique common fixed point in X.

Now we state our second main result which uses the concept of joint reciprocal continuity.

Theorem 2.7: Let P, S and T be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)$ $d(Px, Py, a) \leq \varphi(\{ d(Py, Ty, a)(2 + d(Px, Sx, a)), \ d(Px, Sx, a)(2 + d(Py, Ty, a)\})$

$\forall x, y, a \in X$ and for $some$ $\varphi \in \Phi$,

$ii)$ S and T are $continous$

$iii)$ d is $continous$.

$iv)$ (S, T) is $joint$ $reciprocal$ $continous$ $w.r.t.$ P in X.

Then P, S and T have a unique common fixed point in X.

Proof: From $condition(iv)$ there exists a sequence $\{x_n\}$ in X $such$ $that$

$Sx_n = Px_n = Tx_n = z$ for $some$ $z \in X$ (2.5.1)

and

$d(PSx_n, SPx_n, a) = 0 = d(PTx_n, TPx_n, a)$ $\forall a \in X$ (2.5.2)

Applying $condition$ (ii) in equation (2.5.1) and using this in the equation (2.5.2),we get

$PSx_n \to Sz$ and $PTx_n \to Tz$. (2.5.3)

By taking $x = Sx_n$, $y = Tx_n$ in (i) and letting $n \to \infty$, using conditions (ii) , (iii) and (2.5.2) and (2.5.3), we get $Sz = Tz$. (as derived in theorem 2.1)

Similarly taking $x = Tx_n$ and $y = z$ in (i) and letting $n \to \infty$, we get

$Pz = Tz$ (as derived in theorem 2.1).Therefore $Sz = Pz = Tz$.

Taking $x = x_n$ $and\ y = z\ in\ (i)$ and letting $n \to \infty$, we get $z = Pz$.

This is $z\ is\ a\ fixed\ point\ of\ P$. Hence z is a common fixed point of $S, T\ and\ P\ in\ X$. Suppose if x is common fixed point of $S, T\ and\ P\ in\ X$. Then it can be easily proved that $x = z$. Hence z is a unique common fixed point of $S, T\ and\ P\ in\ X$.

Now we have the following corollaries of theorem 2.6.

Corollary 2.8 : Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: condition (i) of corollary (2.2) and conditions (ii), $(iii)\ and\ (iv)$ of theorem (2.7).

Corollary 2.9: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: condition (i) of corollary (2.3) and conditions (ii), $(iii)\ and\ (iv)$ of theorem (2.7).

Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.10: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: conditions (i) of corollary(2.4) and also conditions (ii), $(iii)\ and\ (iv)$ of theorem (2.6).

Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.11: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: conditions (i) of corollary (2.5) and also conditions (ii), $(iii)\ and\ (iv)$ of theorem (2.7).

. Then $P, S\ and\ T$ have a unique common fixed point in X.

Corollary 2.12: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

conditions (i) of corollary 2.7 and also the conditions (ii), $(iii)\ and\ (iv)$ of theorem (2.7).

Then $P, S\ and\ T$ have a unique common fixed point in X.

Example 2.11:

Let $X = R \times R\ \ for\ A, B \in X$, denote the Euclidean distance $A\ and\ B\ by\ |A - B|$.

Define $d: X^3 \to R\ by\ d(A, B, C) = \{|A - B|, |B - C|, |C - A|\}$. Then (X, d)

is a complete 2-metric space. Let $A_0 \in X$. Define the mappings

$P, S\ and\ T\ on\ X$ as $P(A) = A_0\ \forall A \in X\ and\ S = T = I$.

$Define\ \varphi(t) = \eta\, t, 1 < \eta < 12\ then\ \varphi \in \Phi$. Then $P, S\ and\ T$ satisfy all

the properties of Theorem 2.1 and Theorem 2.7 and A_0 is the unique common

fixed point of $P, S\ and\ T\ in\ X$.

References

1) Badshah,V.H. and Gopal Meena: Common fixed point for compatible mappings on 2-metric space, Journal of Indian Acad. Math.Vol 31, No.1 (2009), 23-30.

2) Gähler,S.: 2-metrische Räume und ihre toplogische structure, Math.Nachr.26(1963-64),115-148.

3) Iseki,K: Fixed point theorem in 2-metric spaces . Math. Sem. Notes 3(1975),133-136.

4) Jungck,G.: Compatible mappings and common fixed points, Internat. J.Math. and Math.Sci. 9(1986), 771-779.

5) Rhoades,B.: Contraction type mappings on a 2-mertic space, Math.Nach, 91(1979),151-155.