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Fixed Point Theorems on Neutrosophic Quasi Metric Spaces with Application

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Abstract:

In this paper, we prove Neutrosophic quasi-metric version of the Banach contraction principle which extends the famous Grabiec fixed point theorem. By using this result, we show the existence of fixed point for contraction mappings on the domain of words and apply this approach to deduce the existence of solutions for some recurrence equations associated with the analysis of Quick sort algorithms and divide and Conquer algorithms, respectively..

Keywords **—Quasi-metric space, Common fixed point, G-complete, Metric space, B-contraction.**

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I. INTRODUCTION

Fuzzy set theory was first introduced by Zadeh [20] in 1965 to describe the situations where data are uncertain. Thereafter the concept of fuzzy sets was generalized as intuitionistic fuzzy set by Atanassov [2] in 1984, has a wide range of applications in various fields. With the help of continuous t-norm the concept of fuzzy metric space was modified by Kramosil and Michalek [9] and George and Veeramani [3]. The concept of fuzzy quasi-metric space was introduced by Gregori and Romaguera [4] by generalizing the concept of fuzzy metric space given by Kramosil and Michalek [9].

The concept of intuitionistic fuzzy quasi-metric space was introduced by Tirado [17] by generalizing the notion of intuitionistic fuzzy metric space given by Alaca, Turkoglu and Yildiz [1] to the quasi-metric setting and gave intuitionistic fuzzy quasi-metric version of the Banach contraction principle.

In 1998, Smarandache [13] characterized the new idea called neutrosophic set. Recently, Kirisci et al [8] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space. In 2020, Sowndrarajan and Jeyaraman et al [15] proved some fixed point results in neutrosophic metric spaces. Our basic references are [5], [6], [7], [10], [16], [19].

In this paper, we prove Banach fixed point theorem in Neutrosophic quasi-metric space. The existence of a solution for a recurrence equation which appears in the average case analysis of Quicksort algorithms is obtained as an application.

We generalize the results of Romaguera, Sapena and Tirado [12] and also generalize several known results

II. PRILIMINARIES

Definition 2.1. A 7- tuple $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is said to be an Neutrosophic Quasi Metric Space [*NQMS*] if Ξ is an arbitrary set, $*$ is a continuous t-norm $\diamond$ and $\odot$ are continuous t-conorm and $\mathfrak{R}, \mathfrak{S}, \mathfrak{T}$ are fuzzy sets on $\Xi \times \Xi \times [0,\infty)$ satisfying the following conditions :

(1) $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) + \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) + \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \le 3$
(2) $0 \le \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \le 1; 0 \le \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \le 1$ *and* $0 \le \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \le 1$,
(3) $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, 0) = 0$,
(4) $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{R}(\tilde{\varsigma}, \mathfrak{k}, \varrho) = 1 \Leftrightarrow \mathfrak{k} = \tilde{\varsigma}$ for all $\varrho > 0$
(5) $\mathfrak{R}(\mathfrak{k}, \mathfrak{z}, \varrho + \rho) \ge \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) * \mathfrak{R}(\tilde{\varsigma}, \mathfrak{z}, \rho)$ for all $\mathfrak{k}, \tilde{\varsigma}, \mathfrak{z} \in \Xi$ and $\varrho, \rho > 0$
(6) $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho): [0,\infty) \to [0,1]$ is left continuous,
(7) $\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, 0) = 1$,
(8) $\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{S}(\tilde{\varsigma}, \mathfrak{k}, \varrho) = 0 \Leftrightarrow \mathfrak{k} = \tilde{\varsigma}$ for all $\varrho > 0$
(9) $\mathfrak{S}(\mathfrak{k}, \mathfrak{z}, \varrho + \rho) \le \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \diamond \mathfrak{S}(\tilde{\varsigma}, \mathfrak{z}, \rho)$ for all $\mathfrak{k}, \tilde{\varsigma}, \mathfrak{z} \in \Xi$ and $\varrho, \rho > 0$
(10) $\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho): [0,\infty) \to [0,1]$ is right continuous,
(11) $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = 1$,
(12) $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{T}(\tilde{\varsigma}, \mathfrak{k}, \varrho) = 0 \Leftrightarrow \mathfrak{k} = \tilde{\varsigma}$ for all $\varrho > 0$
(13) $\mathfrak{T}(\mathfrak{k}, \mathfrak{z}, \varrho + \rho) \le \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \odot \mathfrak{T}(\tilde{\varsigma}, \mathfrak{z}, \rho)$, for all $\mathfrak{k}, \tilde{\varsigma}, \mathfrak{z} \in \Xi$ and $\varrho, \rho > 0$
(14) $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho): [0,\infty) \to [0,1]$ is right continuous.

In this case, we say that $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is a Neutrosophic Quasi-Metric (*NQM*) on Ξ. If in addition $\mathfrak{R}, \mathfrak{S}, \textit{and } \mathfrak{T}$ satisfy $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{R}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$, $\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{S}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$ and $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{T}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$ for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$ and $\varrho > 0$ then $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is called a Neutrosophic Metric (*NM*) on Ξ and $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is called a Neutrosophic metric space (*NMS*).

Example 2.2. Let $(\Xi, \tilde{\delta})$ be a quasi-metric space. Define t-norm $n * m = \min\{n, m\}$, t-conorm $n \diamond m = \max\{n, m\}$ and t-conorm $n \odot m = \max\{n, m\}$ and for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$ and $\varrho > 0$,

$\mathfrak{R}_{\tilde{\delta}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\varrho}{\varrho + \tilde{\delta}(\mathfrak{k}, \tilde{\varsigma})}$, $\mathfrak{S}_{\tilde{\delta}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\tilde{\delta}(\mathfrak{k}, \tilde{\varsigma})}{\varrho + \tilde{\delta}(\mathfrak{k}, \tilde{\varsigma})}$ and $\mathfrak{T}_{\tilde{\delta}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\tilde{\delta}(\mathfrak{k}, \tilde{\varsigma})}{\varrho}$

Then $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is a $NQMS$. We call this *NQM* $(\mathfrak{R}, \mathfrak{S}, \mathfrak{T})$ induced by the metric $\tilde{\delta}$ the standard *NQM*. Furthermore it is easy to check that $(\mathfrak{R}_{\tilde{\delta}})^{-1} = \mathfrak{R}_{\tilde{\delta}^{-1}}$ $(\mathfrak{R}_{\tilde{\delta}})^{i} = \mathfrak{R}_{\tilde{\delta}^{i}}$, $(\mathfrak{S}_{\tilde{\delta}})^{-1} = \mathfrak{S}_{\tilde{\delta}^{-1}}$, $(\mathfrak{S}_{\tilde{\delta}})^{i} = \mathfrak{S}_{\tilde{\delta}^{i}}$, $(\mathfrak{T}_{\tilde{\delta}})^{-1} = \mathfrak{T}_{\tilde{\delta}^{-1}}$, $(\mathfrak{T}_{\tilde{\delta}})^{i} = \mathfrak{T}_{\tilde{\delta}^{i}}$. The topology $\mathfrak{I}_{\tilde{\delta}}$ generated by $\tilde{\delta}$ coincides with the topology $\mathfrak{I}_{\mathfrak{R}\mathfrak{S}\mathfrak{T}_{\tilde{\delta}}}$ generated by the induced *NQM* $(\mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$

Remark 2.3. It is clear that if $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is an *NQM* -space then $(\Xi, \mathfrak{R}, *)$ is a fuzzy quasi-metric space. Conversely if $(\Xi, \mathfrak{R}, *)$ is a fuzzy quasi-metric space on Ξ, then $(\Xi, \mathfrak{R}, 1 - \mathfrak{R}, \frac{1}{\mathfrak{R}} - 1, *, \diamond, \odot)$ is an *NQM* -space where $n \diamond m = 1 - [(1 - n) * (1 - m)]$ for all $n, m \in [0, 1]$. If $(\mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$ is an *NQM* on Ξ, then $(\mathfrak{R}^{-1}, \mathfrak{S}^{-1}, \mathfrak{T}^{-1}, *, \diamond, \odot)$ is also an *NQM* on Ξ where $\mathfrak{R}^{-1}$, $\mathfrak{S}^{-1}$ and $\mathfrak{T}^{-1}$ are the fuzzy sets in $\Xi \times \Xi \times (0, \infty)$ defined by $\mathfrak{R}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{R}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$, $\mathfrak{S}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{S}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$ and $\mathfrak{T}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \mathfrak{T}(\tilde{\varsigma}, \mathfrak{k}, \varrho)$.

Moreover if we denote $\mathfrak{R}^{i}$, $\mathfrak{S}^{s}$ and $\mathfrak{T}^{k}$ the fuzzy sets on $\Xi^2 \times [0, \infty)$ given by $\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \min\{\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho), \mathfrak{R}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho)\}$, $\mathfrak{S}^{s}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \max\{\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho), \mathfrak{S}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho)\}$ and $\mathfrak{T}^{k}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \max\{\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho), \mathfrak{T}^{-1}(\mathfrak{k}, \tilde{\varsigma}, \varrho)\}$. Then $(\mathfrak{R}^{i}, \mathfrak{S}^{s}, \mathfrak{T}^{k}, *, \diamond, \odot)$ is an *NM* on Ξ. In order to construct a suitable topology on an *NQM* -space $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ it seems natural to consider balls $\mathfrak{B}(\mathfrak{k}, r, \varrho)$ defined similarly to Park [11] and Alaca, Turkoglu and Yildiz [1] by $\mathfrak{B}(\mathfrak{k}, r, \varrho) = \{\tilde{\varsigma} \in \Xi: \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) > 1 - r, \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) < r$ and $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) < r$ for all $\mathfrak{k} \in \Xi\}$ $r \in (0, 1)$ and $\varrho > 0$. Then one can prove as in park [11] that the family of sets of the form $\{\mathfrak{B}(\mathfrak{k}, r, \varrho) : \mathfrak{k} \in \Xi, 0 < r < 1, \varrho > 0\}$ is a base for the topology $\mathfrak{I}_{\mathfrak{R},\mathfrak{S},\mathfrak{T}}$ on Ξ.

Definition 2.4. Let $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ be a NMS. A sequence $\{\mathfrak{k}_n\}_n$ in Ξ is called a Cauchy if for each $\varepsilon \in (0, 1)$ and each $\varrho > 0$, there exists $n_0 \in \mathfrak{S}$ such that $\mathfrak{R}(\mathfrak{k}_n, \mathfrak{k}_m, \varrho) > 1 - \varepsilon$, $\mathfrak{S}(\mathfrak{k}_n, \mathfrak{k}_m, \varrho) < \varepsilon$ and $\mathfrak{T}(\mathfrak{k}_n,$

$\mathfrak{k}_m, \varrho) < \varepsilon$ whenever $n, m \geq n_0$. We say that $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is complete if every Cauchy sequence is convergent.

Definition 2.5. A sequence $\{x_n\}_n$ in an *NMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is said to be converges to a point $\mathfrak{k} \in \Xi$ if and only if
$lim_{n\to\infty} \mathfrak{R}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 1, lim_{n\to\infty} \mathfrak{S}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 0$ *and* $lim_{n\to\infty} \mathfrak{T}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 0$, for all $\varrho > 0$.

Definition 2.6. Let $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ be an *NMS*. A sequence $\{\mathfrak{k}_n\}_n$ in Ξ is called G-Cauchy if for each $p \in \mathfrak{S}$ and each $\varrho > 0$, $lim_{n\to\infty} \mathfrak{R}(\mathfrak{k}_n, \mathfrak{k}_{n+p}, \varrho) = 1$, $lim_{n\to\infty} \mathfrak{S}(\mathfrak{k}_n, \mathfrak{k}_{n+p}, \varrho) = 0$ and $lim_{n\to\infty} \mathfrak{T}(\mathfrak{k}_n, \mathfrak{k}_{n+p}, \varrho) = 0$. We say that $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$ is G-complete if every G-Cauchy sequence is convergent.

III. BANACH CONTRACTION IN NEUTROSOPHIC QUASI METRIC SPACES

Definition 3.1. A B-contraction on an *NMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is a self mapping f on Ξ such that there is a constant $\boldsymbol{k} \in (0, 1)$ satisfying
$\mathfrak{R}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \geq \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$, $\mathfrak{S}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ *and* $\mathfrak{T}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ *for all* $\mathfrak{k}, \tilde{\varsigma} \in \Xi$, $\varrho > 0$.

Definition 3.2. A sequence $\{\mathfrak{k}_n\}_n$ in an *NQMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is said to be G-Cauchy if it is a G-Cauchy sequence in the *NMS* $(\Xi, \mathfrak{R}^i, \mathfrak{S}^s, \mathfrak{T}^k, *, \diamond, \odot)$.

Definition 3.3. An *NQMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is called G-bicomplete if the *NMS* $(\Xi, \mathfrak{R}^i, \mathfrak{S}^s, \mathfrak{T}^k, *, \diamond, \odot)$ is G-complete. In this case we say that $(\mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is a fuzzy quasi-metric on Ξ.

Definition 3.4. A B-contraction on an $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ is a self-mapping f on Ξ such that there is a constant $k \in (0, 1)$ satisfying

$$\mathfrak{R}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \geq \mathfrak{R}(k, \tilde{\varsigma}, \varrho), \mathfrak{S}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \text{ and }$$
$$\mathfrak{T}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \text{ for all } \mathfrak{k}, \tilde{\varsigma} \in \Xi, \varrho > 0.$$

The number k is called a contraction constant of f.

Theorem 3.5. Let $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T} *, \diamond, \odot)$ be a G-bicomplete NQMS such that
$\lim_{n\to\infty} \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = 1$, $\lim_{n\to\infty} \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = 0$ and $\lim_{n\to\infty} \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = 0$, for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$.
Then every B-contraction on Ξ has a unique fixed point.

Proof. Let $f : \Xi \longrightarrow \Xi$ be a B-contraction on Ξ with contraction constant $k \in (0, 1)$. Then $\mathfrak{R}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \geq \mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$, $\mathfrak{S}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ and $\mathfrak{T}(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$, $\varrho > 0$.
It immediately follows that
$\mathfrak{R}^i(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \geq \mathfrak{R}^i(\mathfrak{k}, \tilde{\varsigma}, \varrho)$, $\mathfrak{S}^s(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{S}^s(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ and $\mathfrak{T}^k(f(\mathfrak{k}), f(\tilde{\varsigma}), k\varrho) \leq \mathfrak{T}^k(\mathfrak{k}, \tilde{\varsigma}, \varrho)$ for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$, $\varrho > 0$.
Hence f is a B-contraction on the G-complete fuzzy metric space $(\Xi, \mathfrak{R}^i, \mathfrak{S}^s, \mathfrak{T}^k, *, \diamond, \odot)$ and f has a unique fixed point.

IV. NON-ARCHIMEDEAN IN NEUTROSOPHIC QUASI METRIC SPACES

Definition 4.1. An *NQMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$ is called a non-Archimedean *NQMS* if $(\mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$ is a non-Archimedean *NQM* on Ξ, that is,

$$\mathfrak{R}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \geq \min\{\mathfrak{R}(\mathfrak{k}, \mathfrak{z}, \varrho), \mathfrak{R}(\mathfrak{z}, \tilde{\varsigma}, \varrho)\},$$
$$\mathfrak{S}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \leq \max\{\mathfrak{S}(\mathfrak{k}, \mathfrak{z}, \varrho), \mathfrak{S}(\mathfrak{z}, \tilde{\varsigma}, \varrho)\}$$

and $\mathfrak{T}(\mathfrak{k}, \tilde{\varsigma}, \varrho) \leq \max\{\mathfrak{T}(\mathfrak{k}, \mathfrak{z}, \varrho), \mathfrak{T}(\mathfrak{z}, \tilde{\varsigma}, \varrho)\}$, for all $\mathfrak{k}, \tilde{\varsigma}, \mathfrak{z} \in \Xi$ and $\varrho > 0$.

Lemma 4.2. *Each G-Cauchy sequence in a non-Archimedean NQMS is a Cauchy sequence.*

Proof. Let $(\mathfrak{k}_n)$ be G-Cauchy sequence in the non-Archimedean *NQMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$, then for each $\varrho > 0$, we have $\lim_{n\to\infty} \mathfrak{R}^i(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) = 1$, $\lim_{n\to\infty} \mathfrak{S}^s(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) = 0$ and $\lim_{n\to\infty} \mathfrak{T}^k(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) = 0$, which implies that, for each $\varepsilon \in (0, 1)$, there is $n_0 \in \mathbb{N}$ such that

$\mathfrak{R}^{i}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) > 1-\varepsilon$, $\mathfrak{S}^{s}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) < \varepsilon$ and $\mathfrak{T}^{k}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho) < \varepsilon$ for each $n \geq n_0$.

Now let $m > n \geq n_0$. Then $m = n + j$, for some $j \in \mathbb{N}$. So,

$$\mathfrak{R}^{i}(\mathfrak{k}_n, \mathfrak{k}_m, \varrho) \geq \min\{\mathfrak{R}^{i}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho), \mathfrak{R}^{i}(\mathfrak{k}_{n+1}, \mathfrak{k}_{n+2}, \varrho) \ldots, \mathfrak{R}^{i}(\mathfrak{k}_{n+j-1}, \mathfrak{k}_{n+j}, \varrho)\} > 1-\varepsilon,$$

$$\mathfrak{S}^{s}(\mathfrak{k}_n, \mathfrak{k}_m, \varrho) \leq \max\{\mathfrak{S}^{s}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho), \mathfrak{S}^{s}(\mathfrak{k}_{n+1}, \mathfrak{k}_{n+2}, \varrho) \ldots, \mathfrak{S}^{s}(\mathfrak{k}_{n+j-1}, \mathfrak{k}_{n+j}, \varrho)\} < \varepsilon$$

And

$$\mathfrak{T}^{k}(\mathfrak{k}_n, \mathfrak{k}_m, \varrho) \leq \max\{\mathfrak{T}^{k}(\mathfrak{k}_n, \mathfrak{k}_{n+1}, \varrho), \mathfrak{T}^{k}(\mathfrak{k}_{n+1}, \mathfrak{k}_{n+2}, \varrho) \ldots, \mathfrak{T}^{k}(\mathfrak{k}_{n+j-1}, \mathfrak{k}_{n+j}, \varrho)\} < \varepsilon$$

We conclude that $(\mathfrak{k}_n)$ is a Cauchy sequence in $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$.

Theorem 4.3. *Each bicomplete non-Archimedean NQMS is G-bicomplete.*

Proof. Let $(\mathfrak{k}_n)$ be a G-Cauchy sequence in the bicomplete non-Archimedean *NQMS* $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$. By Lemma 4.2, $(\mathfrak{k}_n)$ is a Cauchy sequence in $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$. Hence there is $\mathfrak{k} \in \Xi$ such that $lim_{n\to\infty} \mathfrak{R}^{i}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 1$, $lim_{n\to\infty} \mathfrak{S}^{s}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 0$ and $lim_{n\to\infty} \mathfrak{T}^{k}(\mathfrak{k}, \mathfrak{k}_n, \varrho) = 0$ for all $\varrho > 0$. We conclude that $(\Xi, \mathfrak{R}^{i}, \mathfrak{S}^{s}, \mathfrak{T}^{k}, *, \diamond, \odot)$ is G-complete, that is, $(\Xi, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, *, \diamond, \odot)$ is G-bicomplete.

Corollary 4.4. *Each complete non-Archimedean NMS is G-complete.*

V. APPLICATION

Let Σ be a non-empty alphabet. Let Σ^{∞} be the set of all finite and infinite sequences over Σ, where we adopt the convention that the empty sequence Φ is an element of Σ^{∞}. The symbol $\sqsubseteq$ denote the prefix order on Σ^{∞}, that is, $\mathfrak{k} \sqsubseteq \tilde{\varsigma} \Leftrightarrow \mathfrak{k}$ is a prefix of $\tilde{\varsigma}$. Now, for each $\mathfrak{k} \in \Sigma^{\infty}$ denote by $l(\mathfrak{k})$ the length of $\mathfrak{k}$. Then $l(\mathfrak{k}) \in [1, \infty)$ whenever $\mathfrak{k} \neq \Phi$ and $l(\Phi) = 0$. For each $\mathfrak{k}, \tilde{\varsigma} \in \Sigma^{\infty}$ let $\mathfrak{k} \sqcap \tilde{\varsigma}$ be the common prefix of $\mathfrak{k}$ and $\tilde{\varsigma}$. Thus the function $\tilde{\delta}_{\sqsubseteq}$ defined on Σ^{∞} x Σ^{∞} by

$$\tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma}) = \begin{cases} 0, & if\ \mathfrak{k} \sqsubseteq \tilde{\varsigma} \\ 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}, & otherwise \end{cases}$$

is a quasi-metric on Σ^{∞} (We adopt the convention that $2^{-\infty} = 0$). Actually, $\tilde{\delta}_{\sqsubseteq}$ is a non-Archimedean quasi-metric on Σ^{∞} and the non-Archimedean quasi-metric $(\tilde{\delta}_{\sqsubseteq})^{s}$ is the Baire metric on Σ^{∞}, that is, $(\tilde{\delta}_{\sqsubseteq})^{s}(\mathfrak{k}, \mathfrak{k}) = 0$ and $(\tilde{\delta}_{\sqsubseteq})^{s}(\mathfrak{k}, \tilde{\varsigma}) = 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}$ for all $\mathfrak{k}, \tilde{\varsigma} \in \Sigma^{\infty}$ such that $\mathfrak{k} \neq \tilde{\varsigma}$. It is well known that $(\tilde{\delta}_{\sqsubseteq})^{s}$ is complete. From this fact it is clear that $\tilde{\delta}_{\sqsubseteq}$ is bicomplete. The quasi-metric $\tilde{\delta}_{\sqsubseteq}$,which was introduced by Smyth [16], will be called the Baire quasi-metric. Observe that condition $\tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma}) = 0$ can be used to distinguish between the case that $\mathfrak{k}$ is a prefix of $\tilde{\varsigma}$ and the remaining cases.

Example 5.1. Let $\tilde{\delta}_{\sqsubseteq}$ be a (non-Archimedean) quasi-metric on a set Ξ and let $\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}$ *and* $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}$ are fuzzy sets in $\Xi \times \Xi \times [0, \infty)$ given by

$$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\varrho}{\varrho + \tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma})}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma})}{\varrho + \tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma})}$$

$$\text{and} \quad \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho) = \frac{\tilde{\delta}_{\sqsubseteq}(\mathfrak{k}, \tilde{\varsigma})}{\varrho}$$

for all $\mathfrak{k}, \tilde{\varsigma} \in \Xi$ and $\varrho > 0$. Then $(\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}, \wedge, \vee, \nabla)$ is a (non-Archimedean) *NQM* on Ξ, where $\wedge$ denotes the continuous t-norm ,$\vee$ and ∇ denotes the continuous t-conorm given by $n \wedge m = \min\{n, m\}$, $n \vee m = \max\{n, m\}$ and $n \nabla m = \max\{n, m\}$. It is clear that $\mathfrak{I}_{\mathfrak{R}\mathfrak{S}\mathfrak{T}\tilde{\delta}_{\sqsubseteq}} = \mathfrak{I}_{\tilde{\delta}_{\sqsubseteq}}$ and that $(\Xi, \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}, \wedge, \vee, \nabla)$ is bicomplete if and only if $(\Xi, \tilde{\delta}_{\sqsubseteq})$ is bicomplete.

Proposition 5.2. $(\Sigma^{\infty}, \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{T}_{\tilde{\delta}}, \wedge, \vee, \nabla)$ *is a G-bicomplete non-Archimedean NQMS.*

Consequently, Theorem 3.5 can be applied to this useful space.

Proposition 5.3. $(\Sigma^{\infty}, \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}, \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}, \wedge, \vee, \nabla)$ *is a G-bicomplete non-Archimedean NQMS.*

The neutrosophic non-Archimedean quasi-metric $(\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}, \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}, \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}, \wedge, \vee, \nabla)$ is given by

$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}(\mathfrak{k}, \tilde{\varsigma}, 0) = 0$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k}, \tilde{\varsigma}, 0) = 1$ and $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k}, \tilde{\varsigma}, 0) = 1$ for all $\mathfrak{k}, \tilde{\varsigma} \in \Sigma^{\infty}$.

$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 1$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$ and $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$ if $\mathfrak{k}$ is a prefix of $\tilde{\varsigma}$ and $\varrho > 0$,

$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 1 - 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}$,

$\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}$ and

$\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = \frac{2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}}{1-2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}}$ if $\mathfrak{k}$ is not a prefix of $\tilde{\varsigma}$ and $\varrho \in (0, 1)$,

$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq_1}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 1$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$ and $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq_0}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$ if $\mathfrak{k}$ is not a prefix of $\tilde{\varsigma}$ and $\varrho > 1$.

Proposition 5.3 allows us to apply any of the Proposition 5.2 and Theorem 3.5 to the complexity analysis of quicksort algorithm , to show , in direct way, the existence and uniqueness of solution for the following recurrence equation:

$$T(1) = 0 \text{ and } T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1),\ n \geq 2.$$

Consider as an alphabet Σ the set of non-negative real numbers, that is, $\Sigma = [0, \infty)$. We associate to T the functional $\Phi: \Sigma^{\infty} \longrightarrow \Sigma^{\infty}$ given by $(\Phi(\mathfrak{k}))_1 = T(1)$ and $(\Phi(\mathfrak{k}))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}\mathfrak{k}_{n-1}$, for all $n \geq 2$.If $\mathfrak{k} \in \Sigma^{\infty}$ has length $n < \infty$, we write $\mathfrak{k} = \mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3.........\mathfrak{k}_n$, and if $\mathfrak{k}$ is an infinite word we write $\mathfrak{k}=\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3....$ Next we show that Φ is a B-contraction on the G-bicomplete non-Archimedean *NQMS* (Σ^{∞}, $\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}$, $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}$, $\wedge, \vee, \nabla$) with contraction constant $\frac{1}{2}$.

To this end, we first note that, by construction, we have $l(\Phi(\mathfrak{k})) = l(\mathfrak{k}) + 1$ for all $\mathfrak{k} \in \Sigma^{\infty}$ (in particular $l(\Phi(\mathfrak{k})) = \infty$ whenever $l(\mathfrak{k}) = \infty$). Furthermore, it is clear that $\mathfrak{k} \sqsubseteq \tilde{\varsigma} \Leftrightarrow \Phi(\mathfrak{k}) \sqsubseteq \Phi(\tilde{\varsigma})$ and consequently $\Phi(\mathfrak{k} \sqcap \tilde{\varsigma}) \sqsubseteq \Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma})$, for all $\mathfrak{k}$, $\tilde{\varsigma} \in \Sigma^{\infty}$.

Hence $l(\Phi(\mathfrak{k} \sqcap \tilde{\varsigma})) \leq l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma}))$, for all $\mathfrak{k}$, $\tilde{\varsigma} \in \Sigma^{\infty}$.

From the preceding observations we deduce that for all $\mathfrak{k}$, $\tilde{\varsigma} \in \Xi$, if $\mathfrak{k}$ is a prefix of $\tilde{\varsigma}$, then $\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \frac{\varrho}{2}) = \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 1$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \frac{\varrho}{2}) = \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$ and $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \frac{\varrho}{2}) = \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho) = 0$and if $\mathfrak{k}$ is not a prefix of $\tilde{\varsigma}$, then for all $\varrho > 0$.

$$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k})\sqcap\Phi(\tilde{\varsigma}))}} \geq \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k}\sqcap\tilde{\varsigma}))}} \geq \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-(l(\mathfrak{k}\sqcap\tilde{\varsigma})+1)}} \geq \frac{\varrho}{\varrho + 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}} \geq \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho),$$

$$\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{2^{-l(\Phi(\mathfrak{k})\sqcap\Phi(\tilde{\varsigma}))}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k})\sqcap\Phi(\tilde{\varsigma}))}} \leq \frac{2^{-l(\Phi(\mathfrak{k}\sqcap\tilde{\varsigma}))}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k}\sqcap\tilde{\varsigma}))}} \leq \frac{2^{-(l(\mathfrak{k}\sqcap\tilde{\varsigma})+1)}}{\frac{\varrho}{2} + 2^{-(l(\mathfrak{k}\sqcap\tilde{\varsigma})+1)}} \leq \frac{2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}}{\varrho + 2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}} \leq \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho)$$

and

$$\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{2^{-l(\Phi(\mathfrak{k})\sqcap\Phi(\tilde{\varsigma}))}}{\frac{\varrho}{2}} \leq \frac{2^{-l(\Phi(\mathfrak{k}\sqcap\tilde{\varsigma}))}}{\frac{\varrho}{2}} \leq \frac{2^{-(l(\mathfrak{k}\sqcap\tilde{\varsigma})+1)}}{\frac{\varrho}{2}} \leq \frac{2^{-l(\mathfrak{k}\sqcap\tilde{\varsigma})}}{\varrho} \leq \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k},\tilde{\varsigma},\varrho).$$

Therefore, Φ is a B-contraction on (Σ^{∞},$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}$, $\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}$,$\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}$, $\wedge, \vee, \nabla$) with contraction constant $\frac{1}{2}$. So, by Theroem 3.5, Φ has a unique fixed point $\tilde{\omega} = \tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3$ ...., which is obviously the unique solution to the recurrence equation T, that is,$\tilde{\omega}_1 = 0$ and $\tilde{\omega}_n = \frac{2(n-1)}{n} + \frac{n+1}{n}\tilde{\omega}_{n-1}$ for all $n \geq 2$.

Result : 5.4

We conclude the paper by applying our results to the complexity analysis of Divide and Conquer algorithm. Divide and Conquer algorithms solve a problem by recursively splitting it into subproblems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus, the complexity of a Divide and Conquer algorithm typically is the solution to the recurrence equation given by $T(1) = c$ and $T(n) = aT\left(\frac{n}{b}\right) + h(n)$. Where $a, b, c \in \mathbb{N}$ with $a, b \geq 2$, n range over the set $\{b^p : p = 0, 1, 2,...\}$ and $h(n) \geq 0$ for all $n \in \mathbb{N}$. As in the case of Quicksort algorithm, take $\Sigma = [0, \infty)$ and put $\Sigma^N=\{\mathfrak{k} \in \Sigma^{\infty} : l(\mathfrak{k}) = \infty\}$.Clearly Σ^N is a closed subset of (Σ^{∞}, $(\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}})^i$, $(\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}})^s$, $(\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}})^k$, $\wedge, \vee, \nabla$), (Σ^N,$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}$,$\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}$, $\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}$, $\wedge, \vee, \nabla$) is a non-Archimedean neutrosophic G-bicomplete quasi metric space by proposition 5.2.

Now, we associate to T the functional $\Phi : \Sigma^N \to \Sigma^N$ given by $(\Phi(\mathfrak{k}))_1 = T(1)$ and $(\Phi(\mathfrak{k}))_n = \frac{a\mathfrak{k}_n}{b} + h(n)$ if $n \in \{ b^p : p = 1, 2, ... \}$ and $(\Phi(\mathfrak{k}))_n = 0$ otherwise for all $\mathfrak{k} \in \Sigma^N$.

For our purposes here it suffices to observe that for each $\mathfrak{k}$, $\tilde{\varsigma} \in \Sigma^N$, the following inequality holds $l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma})) \geq 1 + l(\mathfrak{k} \sqcap \tilde{\varsigma})$. In fact, If $l(\mathfrak{k} \sqcap \tilde{\varsigma}) = 0$, then $l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma})) \geq 1$ and if $b^p > l(\mathfrak{k} \sqcap \tilde{\varsigma}) \geq b^{p-1}$, $p \geq 1$, then $b^{p+1} > l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma})) \geq b^p$.

Hence, for each $\mathfrak{k}$, $\tilde{\varsigma} \in \Sigma^N$ and $\varrho > 0$, we obtain

$$\mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma}))}} \geq \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k} \sqcap \tilde{\varsigma}))}} \geq \frac{\frac{\varrho}{2}}{\frac{\varrho}{2} + 2^{-(l(\mathfrak{k} \sqcap \tilde{\varsigma})+1)}} \geq \frac{\varrho}{\varrho + 2^{-l(\mathfrak{k} \sqcap \tilde{\varsigma})}} \geq \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho),$$

$$\mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{2^{-l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma}))}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma}))}} \leq \frac{2^{-l(\Phi(\mathfrak{k} \sqcap \tilde{\varsigma}))}}{\frac{\varrho}{2} + 2^{-l(\Phi(\mathfrak{k} \sqcap \tilde{\varsigma}))}} \leq \frac{2^{-(l(\mathfrak{k} \sqcap \tilde{\varsigma})+1)}}{\frac{\varrho}{2} + 2^{-(l(\mathfrak{k} \sqcap \tilde{\varsigma})+1)}} \leq \frac{2^{-l(\mathfrak{k} \sqcap \tilde{\varsigma})}}{\varrho + 2^{-l(\mathfrak{k} \sqcap \tilde{\varsigma})}} \leq \mathfrak{S}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho)$$

and

$$\mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\Phi(\mathfrak{k}), \Phi(\tilde{\varsigma}), \tfrac{\varrho}{2}) = \frac{2^{-l(\Phi(\mathfrak{k}) \sqcap \Phi(\tilde{\varsigma}))}}{\frac{\varrho}{2}} \leq \frac{2^{-l(\Phi(\mathfrak{k} \sqcap \tilde{\varsigma}))}}{\frac{\varrho}{2}} \leq \frac{2^{-(l(\mathfrak{k} \sqcap \tilde{\varsigma})+1)}}{\frac{\varrho}{2}} \leq \frac{2^{-l(\mathfrak{k} \sqcap \tilde{\varsigma})}}{\varrho} \leq \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}(\mathfrak{k}, \tilde{\varsigma}, \varrho).$$

Therefore Φ is a *B*-contraction on $(\Sigma^{N,}, \mathfrak{R}_{\tilde{\delta}_{\sqsubseteq}}, N_{\tilde{\delta}_{\sqsubseteq}}, \mathfrak{T}_{\tilde{\delta}_{\sqsubseteq}}, \wedge, \vee, \nabla)$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.5, Φ has a unique fixed point $\tilde{\omega} = \tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 \ldots$. Consequently, the function $\boldsymbol{F}$ defined on $\{b^p : p = 0, 1, 2 ...\}$ by $F(b^p) = \omega_{b^p}$ for all $p \geq 0$, is the unique solution to the recurrence equation of the given Divide and Conquer algorithm

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