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Transversal hypersurfaces in pseudo-Riemannian manifolds

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Abstract:
In this paper we derive a condition of transversality of two given hypersurfaces in pseudo-Riemnnaian manifolds, along its boundary.This condition is given by the ellipticciy of the Newton transformations.

Keywords **—Newton transformations,Symmetric functions, Transversality.**

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I. PRELIMINARIES

In this section we will recall basic formulas and notions about hyper-surfaces in pseudo-Riemannian space forms that will be used later on.For more details see [7].
LetM^{n+1}an(n+1)dimensionalpseudo-Riemannianmanifoldofindex $q \geq 0$ and let M^nbe a nondegenerate oriented hypersurface of M^{n+1}. If we denote by A the corresponding shape operator, then at each $p \in M^n$,Arestrictstoaself-adjointlinearmap

$A_p: T_pM \to T_pM.$

Associatedto$A_p$thereare$n$algebraicinvariantsdefinedby
$S_r = \sigma_r(x_1(p),\dots,x_n(p))$.
Where

$$\sigma_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

arethe elementary symmetric functions and $x_1(p),\dots,x_n(p)$ are the pricipale curvature of the hypersurface.
For$0 \leq r \leq n$,wedefinether^{th}meancurvatureofthe hypersurfaceby

$$\binom{n}{r} H_r = \varepsilon_N^r S_r = \sigma_r(\varepsilon_N\, x_1, \dots, \varepsilon_N\, x_n).$$

Observe that $H_0 = 0$ and $H_1 = \frac{1}{n} traceA$ is the usual mean curvature of M which is one of the most important extrinsic curvatures of the hypersurface.
Let $\chi(M)$ be the space of vector fields on the manifold M . The classical Newton transformations associated to the shape operator A are defined inductively by

$$\begin{cases} T_0 = 0, \\ T_r = \varepsilon_N^r S_r - \varepsilon_N^{r-1} A T_{r-1} for r \geq 1. \end{cases}$$

Or equivalently by

$$T_r = \varepsilon_N^r S_r - \varepsilon_N^{r-1} S_r A + \cdots . + (-1)^r A^r.$$

where$I$ is the identity maps in$\chi(M)$.

Observe that the operator T_r is self-adjoint and commutes with A. Therefore all basis of T_pM that diagonalizing A at $p$diaginalizes all of the T_r. Let $\{e_1, \dots, e_n\}$ such basis.
Denoting by A_i the restriction of $A$to $\langle e_i \rangle^{\perp}$. It is well know that :

$$\det(tI - A_i) = \sum_{r=0}^{n-1} (-1)^r S_r(A_i) t^{n-1-r}.$$

Where

$$S_r(A_i) = \sum_{\substack{i_1 < \cdots < i_r \\ i_j \neq i}} x_{i_1} \dots x_{i_r}.$$

It is immediate to check that

$$T_r e_i = \varepsilon_N^r S_r(A_i) e_{i.}$$

We refer the reader to [2] and [7] for other details about classical Newton tensors for hypersurfaces in Riemannian and pseuo Riemannian spaces.

II. GENERALIZED NEWTON TRANSFORMATIONSS

Let E be an n-dimensional real vector space and $End(E)$ be the vector space of endomorphisms of E. Denote by $\mathbb{N}$ the set of nonnegative integers and let $\mathbb{N}^2$be the one of multiindex$u = (u_1;\ u_2)$, with $u_i \in \mathbb{N}$.
The length$|u|$ of u is given by $|u| = u_1 + u_2$.
Fo$A = (A_1;\ A_2) \in End(E) \times End(E)$,
$t = (t_1;\ t_2) \in \mathbb{R}^2$and $u \in \mathbb{N}^2$, we set

$$tA = t_1 A_1 + t_2 A_2,$$
$$t^u = {t_1}^{u_1} . {t_2}^{u_2} .$$

The generalized Newton transformations (GNT in brief) is a system of endomorphisms

$$T_u = T_u(A_1;\ A_2),\ \ u \in \mathbb{N}^2$$

that satisfies the following recursive relations [4]

$$T_{(0,0)} = 0,$$
$$T_{(i,j)} = \sigma_{(i,j)} I - A_1 T_{(i-1,j)} - A_2 T_{(i,j-1)}\ ,$$

if$i + j \geq 1$

Where $\sigma_{(i,j)}$are the coe¢cients of the Newton polynomial

$$P_A : \mathbb{R}^q \longrightarrow \mathbb{R}$$

of A given by

$$P_A(t) = \det(I + t_1 A_1 + t_2 A_2)$$
$$= \sum_{i+j<n} \sigma_{(i,j)}\, {t_1}^i . {t_2}^j .$$

III. MAIN RESULTS

Let M^{n+1} αν (ν +1) dimentional connected pseudo−Riemannian manifold of index θ, and $\varphi : M^n \longrightarrow M^{n+1}$ be an oriented connected hypersurface of M^{n+1} with smooth boundary ≅M. Assume the boundary $\Sigma = \varphi(\partial M)$ is a one codimensionsubmanifold of an oriented connected hypersurfaces$P^n \subset M^{n+1}$.
Consider the second fundamental operators, $A_\Sigma, A^\zeta and A^N$ corresponding to the inclusions $\Sigma \subset P^n, P^n \subset M^{n+1}$, $M^n \subset M^{n+1}$respectively.
Where ζ and N are the unit normal vector fields of the inclusions $P^n \subset M^{n+1}$ and $M^n \subset M^{n+1}$ respectivly
Following [2] we consider a local orthogonal frame $\{e_1, \dots, e_{n-1}\}$in _Σ, ν the out pointing conormal unit vector field of Σ and η the unitary vectorfield normal to_Σin P^n.
We have

$$\overline{\nabla}_{e_i} e_j = \sum_{k=0}^{n-1} \varepsilon_k \langle \overline{\nabla}_{e_i} e_j, e_k \rangle e_k + \varepsilon_\nu \langle \overline{\nabla}_{e_i} e_j, \nu \rangle \nu + \varepsilon_N \langle A^N e_i, e_j \rangle N.$$

And

$$\overline{\nabla}_{e_i} e_j = \sum_{k=0}^{n-1} \varepsilon_k \langle \overline{\nabla}_{e_i} e_j, e_k \rangle e_k + \varepsilon_\eta \langle A_\Sigma e_i, e_j \rangle \eta + \varepsilon_\zeta \langle A^\zeta e_i, e_j \rangle \zeta.$$

Thus

$$\varepsilon_\nu \langle \overline{\nabla}_{e_i} e_j, \nu \rangle \nu + \varepsilon_N \langle A^N e_i, e_j \rangle N = \varepsilon_\eta \langle A_\Sigma e_i, e_j \rangle \eta + \varepsilon_\zeta \langle A_\mathrm{P} e_i, e_j \rangle \zeta.$$

Hence

$$\langle A^N e_i, e_j \rangle = \varepsilon_\eta \langle A_\Sigma e_i, e_j \rangle \langle \eta, N \rangle + \varepsilon_\zeta \langle A_\mathrm{P} e_i, e_j \rangle \langle \zeta, N \rangle.$$

Suppose that $P^n$is totally umbilic, so there exist a smooth fucntionλsuch that

$$A_\mathrm{P} = \lambda I_{n-1},$$

andwe have

$$A\Big|_\Sigma = \rho A_\Sigma + \mu \langle \zeta, N \rangle I_{n-1}.$$

This formula shows that the geometry of the inclusion $\Sigma \subset P^n$ is codded by the couple (A_Σ, I_{n-1}), and the geometry of the inclusion $M^n \subset M^{n+1}$ is given by A.

We will use the Newton trasformations and the generalized Newton transformations :

$$T_\mathrm{r} = T_\mathrm{r}\left(A\Big|_\Sigma\right) and T_{k,l} = T_{k,l}(A_\Sigma, I_{n-1}).$$

and the correspondaingelementary symmetric functions

$$\sigma_\mathrm{r}\left(A\Big|_\Sigma\right) and \sigma_{k,l}(A_\Sigma, I_{n-1}).$$

In this case we have

$$\sigma_\mathrm{r}\left(A\Big|_\Sigma\right) = \sum_{k+l=r} \rho^l.\mu^k \sigma_{k,l}.$$

And

$$\langle T_\mathrm{r} \nu, \nu \rangle = \sum_{k+l=r} \rho^l.\mu^k \sigma_{k,l}$$

The matrix $A$is writing in the basis$\{e_1, \ldots, e_{n-1}, \nu\}$

$$\begin{pmatrix} \gamma_1 & 0 & \ldots & \varepsilon_1\langle A\nu, e_1\rangle \\ 0 & \gamma_2 & \ldots & \vdots \\ & & \gamma_{n-1} & \varepsilon_{n-1}\langle A\nu, e_{n-1}\rangle \\ \varepsilon_1\langle A\nu, e_1\rangle & \ldots & \varepsilon_{n-1}\langle A\nu, e_{n-1}\rangle & \varepsilon_\nu\langle A\nu, \nu\rangle \end{pmatrix}$$

Puting

$$A = \begin{pmatrix} A\Big|_\Sigma & B \\ B^T & c \end{pmatrix}.$$

Where,

$$B = \begin{pmatrix} \varepsilon_1\langle A\nu, e_1\rangle \\ \vdots \\ \varepsilon_{n-1}\langle A\nu, e_{n-1}\rangle \end{pmatrix},$$

and

$$c = \varepsilon_\nu \langle A\nu, \nu \rangle.$$

We have the following results.

Proposition 1.

Let M^{n+1} an $(n+1)$ pseudo-Riemannian manifold, and P^n a totally umbilical hypersurface of M^{n+1} .Denoting by $\Sigma \subset P^n$ a compacthypersurfaceof P^n.

Let $\varphi: M^n \longrightarrow M^{n+1}$ be an oriented connected hypersurfaceof $M^{n+1}$with boundary $\Sigma = \varphi(\partial M)$.

Then along the boundary Σ, we have

$$\langle T_\mathrm{r} \nu, \nu \rangle = \sigma_\mathrm{r}\left(A\Big|_\Sigma\right).$$

Theorem 1.

Under the above hypothesis, M^n and P^n are transversealong Σ if for some $1 \le r \le n$ the Newton transformation $T_\mathrm{r}$is positif defined.

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