International Journal of Scientific Research and Engineering Development–Volume 6 Issue 1, Jan-Feb 2023
Available at www.ijsred.com

The Kumaraswamy-Gull Alpha Power Rayleigh Distribution: Properties and Application to HIV/AIDS Data

Mohamed Kpangay[1 *], Leo O. Odongo[2], George O. Orwa[3]

[1*]Department of Mathematics and Statistics, Njala University, Njala, Sierra Leone
[2]Department of Mathematics and Actuarial Science, Kenyatta University, Nairobi, Kenya
[3]Department of Statistics and Actuarial Science, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

Abstract:
In this paper, a new four-parameter distribution called the Kumaraswamy-Gull Alpha Power Rayleigh distribution (abbreviated as Kw-GAPR) has been proposed. Some well-known sub-models are obtained from the proposed Kw-GAPR distribution. Some statistical properties of the proposed Kw-GAPR distribution are investigated. The distribution parameters are estimated by using the method of maximum likelihood estimation. To assess the performance of the MLE estimates, a Monte Carlo simulation study was conducted. According to the results of the simulation study, as the sample size is increased, average estimates converge to the parameters' true values and average bias, MSE, and RMSE generally decrease. The proposed Kw-GAPR distribution and its sub-models are then compared by fitting the models to an HIV/AIDS data set. From the fitted data set, a conclusion can be drawn that the proposed Kw-GAPR distribution outperforms its competing sub-models.

Keywords - Gull Alpha Power Family, Kumaraswamy Family, Maximum Likelihood Estimation, Rayleigh distribution

I. INTRODUCTION

When it comes to the fundamental concept of statistics that is most widely used in both theory and practice then probability distributions are at the lead. In some fields of study data analysis is the key but these data may exhibit two or more data characteristics such as skewness, kurtosis, monotonic hazard rates, and non-monotonic failure rates at the same time while a majority of fundamental distributions and some families of distributions are unable to accommodate due to the single shape parameter of their cumulative density function. In other to handle the aforementioned data characteristics, mathematicians and statisticians have introduced the idea of developing new distributions that are flexible enough to handle two or more of the data characteristics at the same time by adding extra shape parameter(s) to the classical or fundamental distributions. This addition of extra shape parameters to distributions have been done many ways such as the variable transformation, exponentiation method, Quantile method, the combination of two or more distributions/models method, etc.

However, a large number of novel distributions and families of distributions have been developed and extended with the help of the aforementioned techniques, these novel developed distributions including the exponentiated Rayleigh distribution using Bayesian analysis by [1], Slashed exponentiated rayleigh distribution by [2], the exponentiated rayleigh distribution based on generalized Type-II hybrid censored data by [3], Exponentiated Rayleigh distribution using MCMC approach by [4], the exponentiated generalized class of distributions by [5], a note on Kumaraswamy exponentiated Rayleigh distribution by [6], the Kumaraswamy Marshall-Olkin family by [7], the Kumaraswamy Transmute-G family of distributions by [8], the generalized Marshall-Olkin Kumaraswamy-G family by [9], the Kumaraswamy alpha power-G family by [10] and among others.

In 2020, [11] introduced a new distribution called the Gull Alpha power Weibull distribution which was applied in real and simulated data sets. The Gull alpha Power is a family of distributions with a single shape parameter of its cumulative density function (CDF).
For the past two years, researchers have been extending the Gull Alpha power to make the family flexible by handling two or more of the data characteristics, these extensions including the Gull Alpha Power Chen-G type by [12], the Gull Alpha Power Ampadu-G type by [13], the exponentiated generalized gull alpha power Rayleigh (EGGAPR) distribution by [14], the exponentiated generalized gull alpha power exponential (EGGAPE) distribution by [15], the exponentiated gull alpha power exponential (EGAPE) distribution by [16], An extended Kumaraswamy-Gull Power Exponential (K-GAPE) distribution by [17].
This paper introduces a four-parameterized model known as the Kw-GAPR distribution.

According to [11] the Gull Alpha Power Family (GAPF) of distributions with CDF, $G(x)$ and PDF, $g(x)$ respectively are given as:

$$G(x) = \begin{cases} \frac{\eta G(x)}{\eta^{G(x)}}, & if \quad \eta > 0, \eta \neq 1 \\ G(x), & if \quad \eta = 1 \end{cases} \tag{1}$$

$$g(x) = log(\eta)\eta^{1-G(x)}(-g(x)G(x)) + g(x)\eta^{1-G(x)}, \tag{2}$$

for $\eta > 0, \eta \neq 1$
The GAPF's CDF and PDF are used as a baseline to develop the new family known as the Kamuraswamy-Gull Alpha Power Family, the Kumaraswamy family of distributions, which was introduced by [18]. The main objective of extending this family is to create a distribution that has various shapes (i.e. different hazard rate shapes), gives a distribution that offers a better fit compared to other models with the same baseline distribution, and have heavy-tailed to model various real data sets. The Kumaraswamy-Gull Alpha Power Rayleigh distribution, written as Kw-GAPR, was developed using the Rayleigh distribution as a baseline distribution in the new family of distributions.

The remainder of the article is organized as follows: Section "Development of Kw-GAPR distribution" presents the newly developed family, the proposed distribution, and its sub-models, Section "Statistical properties" presents some statistical properties of the Kw-GAPR distribution, Section "Parameters Estimation" presents the parameters estimation and the Monte Carlo simulation study results. The proposed distribution is then used with its competing sub-models to analyze HIV/AIDS data set in Section "Application", and concluding remarks are presented in Section "conclusion".

II. DEVELOPMENT OF THE Kw-GAPR DISTRIBUTION

A. The Kumaraswamy-Gull Alpha Power Family (Kw-GAPF) of Distributions

The CDF, G(x) of Kumaraswamy family of distributions defined by [18] is given as

$$G_{Kw}(x) = 1 - \{1 - [G(x)]^a\}^b, a > 0, b > 0 \tag{3}$$

with PDF, $g(x)$ as

$$g_{_{Kw}}(x) = abg(x)[G(x)]^{a-1}\{1 - [G(x)]^a\}^{b-1} \tag{4}$$

where $x > 0$, $a > 0$, and $b > 0$

To develop the new family of distributions called Kw-GAPF, we substitute Eqs. (1) and (2) of GAPF into Eqs. (3) and (4). The new family will have a CDF defined by

$$G_{Kw-GAPF}(x) = 1 - \left\{1 - \left[\frac{\eta G(x)}{\eta^{G(x)}}\right]^a\right\}^b \tag{5}$$

and PDF defined by

$$g_{Kw-GAPF}(x) = ab(log(\eta)\eta^{1-G(x)}(-g(x)G(x)) + g(x)\eta^{1-G(x)}) \times \left[\frac{\eta G(x)}{\eta^{G(x)}}\right]^{a-1} \times \left\{1 - \left[\frac{\eta G(x)}{\eta^{G(x)}}\right]^a\right\}^{b-1} \tag{6}$$

for $a > 0, b > 0$.

B. The Kw-GAP Rayleigh distribution

The proposed Kw-GAPR (x, a, b, η, μ) distribution has the CDF, PDF, and $h(x)$ defined as:

$$G(x) = 1 - \left\{1 - \left[\frac{\eta(1 - e^{-\frac{x^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x^2}{2\mu^2}}}}\right]^a\right\}^b \tag{7}$$

$$g(x) = ab\left(\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}} - log(\eta)\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}(1- e^{-\frac{x^2}{2\mu^2}})e^{-\frac{x^2}{2\mu^2}}\right)\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1} \times\left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1} \quad (8)$$

$$h(x) = \frac{M\left[\frac{\eta(1-e^{-\frac{x^2}{2\mu^2}})}{\eta^{e^{-\frac{x^2}{2\mu^2}}}}\right]^{a-1}\left\{1-\left[\frac{\eta(1-e^{-\frac{x^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x^2}{2\mu^2}}}}\right]^a\right\}^{b-1}}{\left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^b} \quad (9)$$

for $M = ab(\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}} - log(\eta)\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}(1 - e^{-\frac{x^2}{2\mu^2}})e^{-\frac{x^2}{2\mu^2}}), x > 0, \mu > 0$.

From **Fig. 1**, the Kw-GAPR density function can be unimodal, J-shaped, right-skewed, virtually symmetric, left-skewed, with a heavy-tail, and highly flexible kurtosis, making it ideal for modeling variety of data sets.

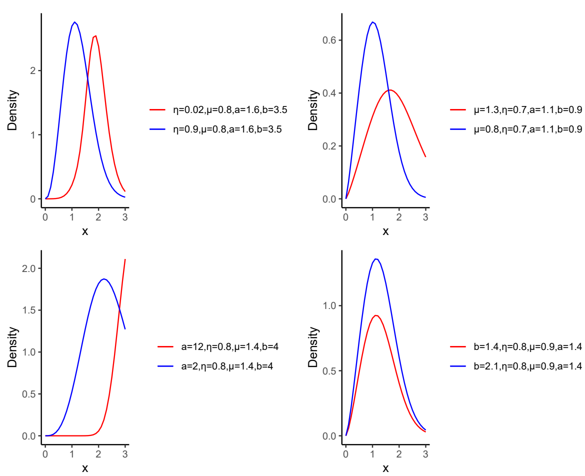


Fig. 1: **Kw-GAPR density shapes for some parameters values**

It is observed from **Fig. 2** that the plots of the hazard rate function for different parameter values show a variety of shapes, which include decreasing, increasing, inverted bathtub shapes, and increasing-decreasing shapes. The Kw-GAPR distribution has very attractive characteristics that make it suitable for modeling monotonic and non-monotonic hazard behaviors that are more likely to be encountered in real-world situations like reliability analysis, human mortality, and biomedical applications, thus increasing its adaptability to fit various survival data.

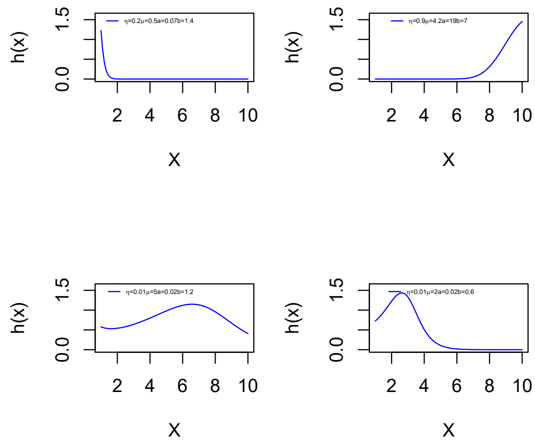


Fig. 2: **Kw-GAPR Hazard rate function shapes**

C. Sub-Models of the Kw-GAPR

The Kw-GAPR distribution have some special sub-models. These sub-models include:

(i) The Rayleigh (R) distribution

When $\eta = a = b = 1$, the Kw-GAPR distribution's CDF reduces to the Rayleigh (R) distribution with CDF defined by $G_R(x) = \left(1 - e^{-\frac{x^2}{2\mu^2}}\right)$, for $\mu > 0, x > 0$.

(ii) The Exponentiated Rayleigh (ER) distribution

When $\eta = b = 1$, the Kw-GAPR distribution's CDF reduces to the Exponentiated Rayleigh (ER) distribution with CDF defined by $G_{ER}(x) = \left(1 - e^{-\frac{x^2}{2\mu^2}}\right)^a$, for $\mu > 0, a > 0, \neq 1, x > 0$.

(iii) The Gull Alpha Power Rayleigh (GAPR) distribution

When $a = b = 1$, the Kw-GAPR distribution's CDF reduces to the Gull Alpha Power Rayleigh (GAPR) distribution with CDF defined by $G_{GAPR}(x) = \left[\frac{\eta(1-e^{-\frac{x^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x^2}{2\mu^2}}}}\right]$, for $\eta > 0, \neq 1, \mu > 0, x > 0$

(iv) The Exponentiated Gull Alpha Power Rayleigh (EGAPR) distribution

When $b = 1$ the Kw-GAPR distribution's CDF reduceS to the Exponentiated Gull Alpha Power Rayleigh (EGAPR) distribution with CDF defined by $G_{EGAPR}(x) = \left[\frac{\eta(1-e^{-\frac{x^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x^2}{2\mu^2}}}}\right]^a$, for $\eta, a > 0, \neq 1$, $\mu > 0, x > 0$

(v) The Kumaraswamy Rayleigh (KR) distribution

When $\eta = 1$, the Kw-GAPR distribution's CDF reduces to the Kumaraswamy Rayleigh (KR) distribution with CDF defined by $G_{KR}(x) = 1 - \left[1 - \left(1 - e^{-\frac{x^2}{2\mu^2}}\right)^a\right]^b$, for $\mu > 0, a, b > 0, \neq 1, x > 0$.
Table 1 displays a summary of Kw-GAPR some special sub-models.

Table 1: **Summary of some special sub-models of the Kw-GAPR distribution.**

η	a	b	**Sub-model**
1	1	1	The R distribution
1	a	1	Th ER distribution
η	1	1	The GAPR distribution
η	a	1	The EGAPR distribution
1	a	b	The KR distribution

III. STATISTICAL PROPERTIES

In this section of the paper, the statistical properties of the proposed Kw-GAPR distribution are investigated.

A. *Quantile function*

The quantile function has been used to compute the median, skewness, kurtosis, and to conducted simulation studies, among other things.
Eq. (10) shows the quantile function (i.e. inverse CDF) of the Kw-GAPR distribution which was obtained by using Mathematica software.

$$Q_x(k) = \mu \cdot \sqrt{log\left\{\frac{log\eta^2}{\left(W(z)(-\frac{\left(1-(1-k)^{\frac{1}{b}}\right)^{\frac{1}{a}}log(\eta)}{\eta}) + log(\eta)\right)^2}\right\}}, \tag{10}$$

for $\eta > 0, \neq 1$
where $productLog$ function $W(z)$ is defined by

$$W(z) = \frac{\sum_{n=1}^{\infty}(-1)^{n-1}n^{n-2}}{(n-1)!}z^n$$

Table 2 displays some quantile function values for different combinations of parameter values. For $k = 0.5$ being the median.

Table 2: **Summary of some Quantile values of the Kw-GAPR distribution.**

k	$\eta = 0.7, \mu = 0.3, a = 1.4, b = 2.6$	$\eta = 0.7, \mu = 0.3, a = 1.8, b = 3.2$
0.1	0.162123	0.199292
0.2	0.212372	0.247268
0.3	0.251508	0.283478
0.4	0.286391	0.315175
0.5	0.319925	0.345275
0.6	0.354168	0.375733
0.7	0.391438	0.408649
0.8	0.435743	0.447548
0.9	0.498250	0.502136

B. *The r^{th} Moments*

Given the general form of the r^{th} Moments as

$$\delta_r' = \int_0^{\infty} x^r g(x)dx \tag{11}$$

where $r = 1, 2, 3, 4, ..., n$ and $g(x)$ is the PDF of any distribution, then by substituting the Kw-GAPR PDF as in Eq. (8) into Eq. (11) we have:

$$\delta_r' = ab(\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}} - log(\eta)\eta^{e^{-\frac{x^2}{2\mu^2}}}\frac{x}{\mu^2}(1- e^{-\frac{x^2}{2\mu^2}})e^{-\frac{x^2}{2\mu^2}})\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1} \times \left\{1 - \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1} dx \tag{12}$$

Necessary representation for the binomial expansion.

$$(1-x)^{r-1} = \sum_{t=0}^{\infty} \begin{pmatrix} r-1 \\ t \end{pmatrix} (1)^{n-1-t}(-x)^t = \sum_{t=0}^{\infty}(-1)^t \begin{pmatrix} r-1 \\ t \end{pmatrix} x^t$$

Using the necessary binomial expansion representation in Eq. (12),
we have

$$\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1} = \left(\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^{a-1} \sum_{s=0}^{\infty}(-1)^s \begin{pmatrix} a-1 \\ s \end{pmatrix} \left(e^{-\frac{x^2}{2\mu^2}}\right)^s$$

$$\left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1} = \sum_{t=0}^{\infty}(-1)^t \begin{pmatrix} b-1 \\ t \end{pmatrix} \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{at}$$

but

$$\left[(1-e^{\frac{x^2}{2\mu^2}-})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{at} = \left(\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^{at} \sum_{z=0}^{\infty}(-1)^z \begin{pmatrix} at \\ z \end{pmatrix} \left(e^{-\frac{x^2}{2\mu^2}}\right)^z$$

By employing the three binomial expansions above, Eq. (12) can now be written as;

$$\delta_r^{'} = (\frac{ab}{\mu^2}) \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{z=0}^{\infty} \int_0^{\infty} x^{r+1}(-1)^{s+t+z} \left(\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^{a+at-1} \begin{pmatrix} a-1 \\ s \end{pmatrix} \begin{pmatrix} b-1 \\ t \end{pmatrix} \begin{pmatrix} at \\ z \end{pmatrix} \left(e^{-\frac{x^2}{2\mu^2}}\right)^{s+z} \left(\eta^{e^{-\frac{x^2}{2\mu^2}}} e^{-\frac{x^2}{2\mu^2}}\right)^{s+z} \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{at} \left(1-log(\eta)(1-e^{-\frac{x^2}{2\mu^2}})\right) dx \quad (13)$$

It is noticeable that Eq. (13) is not in closed form, hence, the moments are acquired by numerical integration using R software.

The first five moments of the Kw-GAPR distribution values for various parameter values are shown in **Table 3.** **I:** $\eta = 0.1$, $\mu = 0.4$, $a = 1.1$, $b = 4$; **II:** $\eta = 2$, $\mu = 0.4$, $a = 1.1$, $b = 4$; **III:** $\eta = 0.1$, $\mu = 0.8$, $a = 1.1$, $b = 4$; **VI:** $\eta = 0.1$, $\mu = 1.4$, $a = 1.6$, $b = 4$; **V:** $\eta = 0.1$, $\mu = 0.4$, $a = 1.1$, $b = 1.3$.
It is observed from **Table 3** that the Kw-GAPR distribution is versatile in both means and variance.
Table 3 also depicts that the Kw-GAPR distribution skewness values can be right-skewed (i.e. $S > 0$) and left-skewed (i.e. $S < 0$) while the kurtosis values depict that the Kw-GAPR can be platykurtic (i.e. $K < 3$), almost mesokurtic (i.e. almost $K = 3$), and leptokurtic (i.e $K > 3$).

Table 3: **Summary of r^{th} moments, skewness (S), and kurtosis (K)**

$\delta_r^{'}$	I	II	III	IV	V
$\delta_1^{'}$	0.4942589	0.1960544	0.9885179	0.5941964	0.6867672
$\delta_2^{'}$	0.2701488	0.0482603	1.0805952	0.3740137	0.5224169
$\delta_3^{'}$	0.1586705	0.0139796	1.2693640	0.2467171	0.4302878
$\delta_4^{'}$	0.0986716	0.0046031	1.5787457	0.1693993	0.3790027
$\delta_5^{'}$	0.0643761	0.0016864	2.0600344	0.1204957	0.3541442
Var.	0.02585689	0.00982298	0.10342757	0.02094425	0.05076775
SD	0.1608008	0.0991109	0.3216016	0.1447213	0.2253170
CV	0.3253371	0.5055277	0.3253371	0.2435580	0.3280835
S	-0.0993212	0.6843949	-0.0993203	-0.1354240	0.1554067
K	2.8564070	3.5005446	2.8563956	3.0737262	3.1035235

C. *Skewness and Kurtosis*

The Bowley skewness and Moors kurtosis coefficients are calculated using quartiles and octiles, respectively, to show the influence or effect of the extra shape parameters on the measure of shape skewness and kurtosis. Bowley skewness and Moors kurtosis are insensitive to outliers and are guaranteed to exist even for distributions without moments. The Bowley skewness based on quartiles defined by [19] is given as:

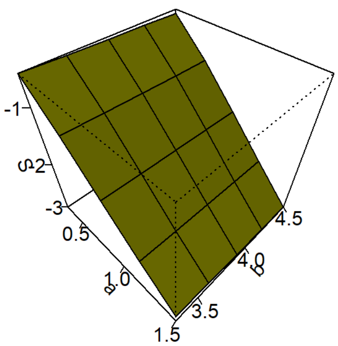
$$S = \frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)}$$

and the Moors kurtosis based on octiles defined by [20] is given as

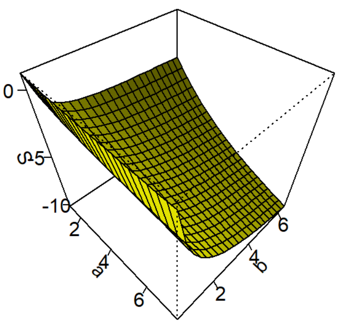
$$K = \frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(0.75)-Q(0.25)}$$

where Q is the quantile function.
The Bowley skewness and the Moors kurtosis are obtained using Eq. (10)

A 3D representation of Bowley's skewness with fixed baseline parameter values of $\eta = 0.3$ and $\mu = 0.7$ is shown in **Fig. 3**. As observed, the extra parameters have an impact on the skewness coefficient's shapes. This increases the adaptability/flexibility of the Kw-GAPR distribution and supports the significance of the additional parameters.



(a)



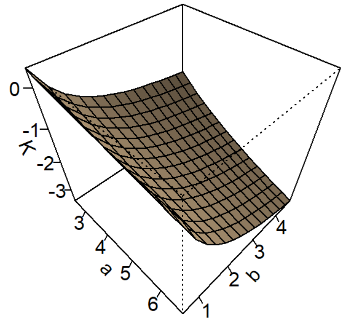
(b)

Fig. 3: **Plots of Bowley's skewness for the Kw-GAPR distribution for fixed baseline parameter values of $\eta = 0.3$ and $\mu = 0.7$.**

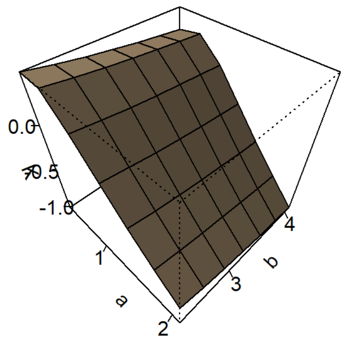
A 3D representation of Moors' kurtosis with fixed baseline parameter values of $\eta = 0.3$ and $\mu = 0.7$ is shown in **Fig. 4**. As observed, the extra parameters have an impact on the kurtosis coefficient's shapes. This increases the adaptability/flexibility of the Kw-GAPR distribution and supports the significance of the additional parameters.

D. Entropy of Kw-GAPR Distribution

In various situations such as in science, engineering, probability theory etc., Entropy has been used as a measure of uncertainty or variation of a random variable.
The Rényi Entropy for a random variable X with any



(a)



(b)

Fig. 4: **Plots of Moors kurtosis for the Kw-GAPR distribution for fixed baseline parameter values of $\eta = 0.3$ and $\mu = 0.7$.**

distribution and order γ defined by [21] is given as

$$R_{\gamma}(X) = \frac{1}{1-\gamma} log \left\{ \int_0^{\infty} \left[g(x) \right]^{\gamma} dx \right\} \qquad (14)$$

for $\gamma > 0, \neq 1, \gamma > 1$.
Substituting Eq. (8) into Eq. (14), we have

$$R_{\gamma}(X) = \frac{1}{1-\gamma} log\{ \int_0^{\infty} (ab(\eta^{e^{-\frac{x^2}{2\mu^2}}} \frac{x}{\mu^2} e^{-\frac{x^2}{2\mu^2}} - log(\eta) \eta^{e^{-\frac{x^2}{2\mu^2}}} \frac{x}{\mu^2} (1 - e^{-\frac{x^2}{2\mu^2}}) e^{-\frac{x^2}{2\mu^2}}) \left[(1 - e^{-\frac{x^2}{2\mu^2}}) \eta^{e^{-\frac{x^2}{2\mu^2}}} \right]^{a-1} \times \left\{ 1 - \left[(1 - e^{-\frac{x^2}{2\mu^2}}) \eta^{e^{-\frac{x^2}{2\mu^2}}} \right]^{a} \right\}^{b-1})^{\gamma} dx \} \qquad (15)$$

Using the necessary binomial representations in Eq. (15), the Rényi entropy of the Kw-GAPR distribution can be written as

$$R_\gamma(X)=\frac{1}{1-\gamma}log[(\frac{ab}{\mu^2})^\gamma\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{z=0}^{\infty}\int_0^\infty x^\gamma \\ \{(-1)^{s+t+z} \\ \left(\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^{a+at-1}\binom{a-1}{s}\binom{b-1}{t}\binom{at}{z} \\ \left(e^{-\frac{x^2}{2\mu^2}}\right)^{s+z}\left(\eta^{e^{-\frac{x^2}{2\mu^2}}}e^{-\frac{x^2}{2\mu^2}}\right)^{s+z} \\ \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{at}\left(1-log(\eta)(1-e^{-\frac{x^2}{2\mu^2}})\right)\}^\gamma dx] \tag{16}$$

Table 4 displays the Kw-GAPR distribution entropy values for different parameter values
I: $\eta=0.7$, $\mu=0.9$, $a=2.9$, $b=1.7$; **II:** $\eta=0.3$, $\mu=1.5$, $a=1.3$, $b=2.5$; **III:** $\eta=0.5$, $\mu=0.4$, $a=1.9$, $b=1.3$. These entropy values were obtained through numerical integration.

Table 4: **Some values of the R$\acute{e}$nyi entropy for the Kw-GAPR distribution.**

R_γ	I	II	III
$R_{(0.3)}$	0.877354	1.320951	0.218472
$R_{(0.7)}$	0.621273	1.112078	-0.026330
$R_{(1.2)}$	0.483811	0.992315	-0.160581
$R_{(1.8)}$	0.395330	0.911272	-0.248079
$R_{(2.4)}$	0.340423	0.859521	-0.302702
$R_{(3.1)}$	0.296916	0.817843	-0.346111

E. *Order Statistic*

The general form of the i^{th} order statistic is given by

$$g_{(i:n)}(x)=\frac{n!}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i}g(x) \tag{17}$$

Therefore, by substituting Eqs. (7) and (8) into Eq. (17), the i^{th} order statistic of the K-GAPE distribution's PDF is given by:

$$g_{(i:n)}(x)=\frac{n!}{(i-1)!(n-i)!}ab(\frac{x}{\mu^2}\eta^{e^{-\frac{x^2}{2\mu^2}}}e^{-\frac{x^2}{2\mu^2}}-\eta^{e^{-\frac{x^2}{2\mu^2}}} \\ log(\eta)\frac{x}{\mu^2}(1-e^{-\frac{x^2}{2\mu^2}})e^{-\frac{x^2}{2\mu^2}})\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1}\times \\ \left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1}\times \\ \left\{1-\left[1-\left((1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{\frac{x^2}{2\mu^2}}}\right)^a\right]^b\right\}^{i-1}\times \\ \left\{\left[1-\left((1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^a\right]^b\right\}^{n-i} \tag{18}$$

while the PDF of the minimum order statistic, $f_{(1:n)}(x)$ is given by

$$g_{(1:n)}(x)=nab\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}}\eta^{e^{-\frac{x^2}{2\mu^2}}}\left(1-log(\eta)(1-e^{-\frac{x^2}{2\mu^2}})\right) \\ \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1}\left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1}\times \\ \left\{\left[1-\left((1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^a\right]^b\right\}^{n-1} \tag{19}$$

and the PDF of the maximum order statistic, $f_{(n:n)}(x)$ is given by

$$g_{(n:n)}(x)=nab\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}}\eta^{e^{-\frac{x^2}{2\mu^2}}}\left(1-log(\eta)(1-e^{-\frac{x^2}{2\mu^2}})\right) \\ \left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^{a-1}\left\{1-\left[(1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right]^a\right\}^{b-1}\times \\ \left\{1-\left[1-\left((1-e^{-\frac{x^2}{2\mu^2}})\eta^{e^{-\frac{x^2}{2\mu^2}}}\right)^a\right]^b\right\}^{n-1} \tag{20}$$

With the minimum and maximum order statistic, the range (R) of the distribution can be calculated by $R_{(i:n)}(x)=g_{(n:n)}(x)-g_{(1:n)}(x)$.

IV. PARAMETERS ESTIMATION

The log-likelihood function is employed in Eq. (8) for the model parameters a, b, η, μ, to determine the MLEs of the given parameter estimation, which is given as

$$log(\ell) = 2nlog(ab) + 2\sum_{i=1}^{n} log(x_i) - 4nlog(\mu) - \frac{1}{\mu^2}\sum_{i=1}^{n}(x_i^2) + log(log(\eta)) + \sum_{i=1}^{n} log(1-e^{-\frac{x_i^2}{2\mu^2}}) + (a-1)\sum_{i=1}^{n} log\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right) + (b-1)\sum_{i=1}^{n} log\left[1-\left(\frac{\eta(1-e^{-\mu x_i^2})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)^a\right] \quad (21)$$

The maximum likelihood estimates ($\hat{a}$, $\hat{b}$, $\hat{\eta}$, $\hat{\mu}$) for the parameters a, b, η, and μ are the values that maximize the log-likelihood function shown in Eq. (21).
The partial derivatives of the log-likelihood function in Eq. (21) with respect to a, b, η, μ are given by:
For ease of differentiation, let

$$j = 2nlog(ab) + 2\sum_{i=1}^{n} log(x_i) - 4nlog(\mu) - \frac{1}{\mu^2}\sum_{i=1}^{n}(x_i^2) + log(log(\eta)) + \sum_{i=1}^{n} log(1-e^{-\frac{x_i^2}{2\mu^2}}),$$

$$m = (a-1)\sum_{i=1}^{n} log\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right),$$

and

$$k = (b-1)\sum_{i=1}^{n} log\left[1-\left(\frac{\eta(1-e^{-\mu x_i^2})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)^a\right]$$

Hence, Eq. (21) can now be written as

$$log(\ell) = j + m + k \quad (22)$$

From Eq. (21), we can express the log-likelihood in terms of j, m, and k (as in Eq. 22) and perform their partial derivatives w.r.t. a, b, η, μ as shown.

$$\frac{\partial j}{\partial a} = \frac{2n}{a}, \frac{\partial j}{\partial b} = \frac{2n}{b}, \frac{\partial j}{\partial \eta} = \frac{1}{\eta \log \eta},$$

$$\frac{\partial j}{\partial \mu} = -\frac{4n}{\mu} + \frac{2}{\mu^3}\sum_{i=1}^{n} x_i^2 - \frac{1}{\mu^3}\sum_{i=1}^{n}\frac{x_i^2 e^{-\frac{x_i^2}{2\mu^2}}}{1-e^{-\frac{x_i^2}{2\mu^2}}}$$

$$\frac{\partial m}{\partial a} = \sum_{i=1}^{n} log\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right), \frac{\partial m}{\partial b} = 0,$$

$$\frac{\partial m}{\partial \eta} = (a-1)\sum_{i=1}^{n}\frac{e^{-\frac{x_i^2}{2\mu^2}}}{\eta},$$

$$\frac{\partial m}{\partial \mu} = (a-1)\sum_{i=1}^{n}\frac{x_i^2 e^{-\frac{x_i^2}{2\mu^2}}[log(\eta)(1-e^{-\frac{x_i^2}{2\mu^2}})-1]}{\mu^3(1-e^{-\frac{x_i^2}{2\mu^2}})}$$

$$\frac{\partial k}{\partial a} = (b-1)\sum_{i=1}^{n}\frac{\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)^a log\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)}{\left[\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)^a - 1\right]},$$

$$\frac{\partial k}{\partial b} = \sum_{i=1}^{n} log\left[1-\left(\frac{\eta(1-e^{-\frac{x_i^2}{2\mu^2}})}{\eta^{1-e^{-\frac{x_i^2}{2\mu^2}}}}\right)^a\right]$$

$$\frac{\partial k}{\partial \eta} = a(b-1)\sum_{i=1}^{n}\frac{e^{-\frac{x_i^2}{2\mu^2}}\left((1-e^{-\frac{x_i^2}{2\mu^2}})\eta^{e^{-\frac{x_i^2}{2\mu^2}}}\right)^a}{\eta\left[\left(\eta^{e^{-\frac{x_i^2}{2\mu^2}}}(1-e^{-\frac{x_i^2}{2\mu^2}})\right)^a - 1\right]}, \frac{\partial k}{\partial \mu} = a(b-1)$$

$$\sum_{i=1}^{n}\frac{\left((1-e^{-\frac{x_i^2}{2\mu^2}})\eta^{e^{-\frac{x_i^2}{2\mu^2}}}\right)^a x_i^2 e^{-\frac{x_i^2}{2\mu^2}}[log(\eta)(1-e^{-\frac{x_i^2}{2\mu^2}})-1]}{\mu^3\left(\left((1-e^{-\frac{x_i^2}{2\mu^2}})\eta^{e^{-\frac{x_i^2}{2\mu^2}}}\right)^a - 1\right)(1-e^{-\frac{x_i^2}{2\mu^2}})}$$

Therefore, the partial derivatives of $\log(\ell)$ for each parameter and equating them to zero are as follows:

$$\frac{\partial log(\ell)}{\partial a} = \frac{\partial j}{\partial a} + \frac{\partial m}{\partial a} + \frac{\partial k}{\partial a} = 0 \quad (23)$$

$$\frac{\partial log(\ell)}{\partial b} = \frac{\partial j}{\partial b} + \frac{\partial m}{\partial b} + \frac{\partial k}{\partial b} = 0 \quad (24)$$

$$\frac{\partial log(\ell)}{\partial \eta} = \frac{\partial j}{\partial \eta} + \frac{\partial m}{\partial \eta} + \frac{\partial k}{\partial \eta} = 0 \quad (25)$$

$$\frac{\partial log(\ell)}{\partial \mu} = \frac{\partial j}{\partial \mu} + \frac{\partial m}{\partial \mu} + \frac{\partial k}{\partial \mu} = 0 \quad (26)$$

From observation, a numerical optimization method is required to find the solutions of Eqs. (23)-(26) due to the fact that they lack closed-form solutions. In this paper, to estimate the Kw-GAPR distribution's parameters, the Broyden-Fletcher-Goldfarb-Shannon (BFGS)

algorithm was utilized, and both the gradient vector of the log-likelihood function and the Hessian matrix is required.The second-ordered partial derivatives of the log-likelihood function with respect to the parameters make up the square matrix known as the Hessian matrix.
The Kw-GAPR distribution's observed information matrix is given by

$$J^{-1}(\varphi) = \begin{bmatrix} \frac{\partial^2 log(\ell)}{\partial\eta^2} & \frac{\partial^2 log(\ell)}{\partial\eta\partial\mu} & \frac{\partial^2 log(\ell)}{\partial\eta\partial a} & \frac{\partial^2 log(\ell)}{\partial\eta\partial b} \\ & \frac{\partial^2 log(\ell)}{\partial\mu^2} & \frac{\partial^2 log(\ell)}{\partial\mu\partial a} & \frac{\partial^2 log(\ell)}{\partial\mu\partial b} \\ & & \frac{\partial^2 log(\ell)}{\partial a^2} & \frac{\partial^2 log(\ell)}{\partial a\partial b} \\ & & & \frac{\partial^2 log(\ell)}{\partial b^2} \end{bmatrix}$$

evaluated at $\hat{\varphi} = (\hat{\eta}, \hat{\mu}, \hat{a}, \hat{b})'$.

Monte Carlo simulation study

To investigate the average biases (ABs), mean square errors (MSEs), root mean square errors (RMSEs), and the average estimates (AEs) of the maximum likelihood estimators for the parameters of the Kw-GAPR distribution a Monte Carlo simulation study was conducted.
Steps involved in conducting a Monte Simulation study:
Step 1. A random sample of sizes **n**=100, 200, ..., 500 with 1000 iterations/replications in each **n** value was generated from the Kw-GAPR distribution using the quantile function given in Eq. (10)
Step 2. Two selected sets of initial values for parameters were used:
Set **I** (η, μ, a, b)=(0.5, 0.3, 1.5, 1.3), and set **II**: (η, μ, a, b)=(0.6, 0.8, 1.6, 1.8).
Step 3. In each set, the simulation results for the average estimates (AEs), average biases (ABs), mean square errors (MSEs), and root mean square errors (RMSEs) are recorded shown in Tables 5 and 6.
To calculate the Average Biases (ABs), the Mean Squared Errors (MSEs), and the Root Mean Squared Errors (RMSEs) the following formulae were used

$$AB_{(\Theta)} = \frac{1}{Z}\sum_{i=1}^{Z}(\widehat{\Theta}_i - \Theta) \tag{27}$$

$$MSE_{(\Theta)} = \frac{1}{Z}\sum_{i=1}^{Z}(\widehat{\Theta}_i - \Theta)^2 \tag{28}$$

and

$$RMSE_{(\Theta)} = \sqrt{\frac{1}{Z}\sum_{i=1}^{Z}(\widehat{\Theta}_i - \Theta)^2} \tag{29}$$

Where Z is the number of iterations, and $\widehat{\Theta}_i$ is an estimator of Θ.
The simulation results for the Kw-GAPR distribution show that the average estimates (AEs) approach the true values of the parameters as the sample size increases as shown in Tables 5 and 6 . The ABs, MSEs, and RMSEs for the estimators of the parameters decrease, in general, as the sample size presented increases (shown in Tables 5 and 6).

Table 5: **Monte Carlo simulation study results for set I parameters**

(η, μ, a, b)=(0.5, 0.3, 1.5, 1.3)								
	AEs				**ABs**			
n	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$
100	0.6485	0.3508	1.6749	2.7632	0.1485	0.0508	0.1749	1.4632
200	0.5838	0.3168	1.5156	2.5967	0.0838	0.0168	0.0156	1.2967
300	0.5767	0.3017	1.4993	2.7489	0.0767	0.0017	-0.0007	1.4489
400	0.5439	0.3045	1.4561	2.6222	0.0439	0.0045	-0.0439	1.3222
500	0.5364	0.3015	1.4637	2.4308	0.0364	0.0015	-0.0363	1.1308
	MSEs				**RMSEs**			
n	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$
100	0.3536	0.0497	0.9326	37.2979	0.5947	0.2229	0.9657	6.1072
200	0.2062	0.0375	0.3049	13.1912	0.4542	0.1937	0.5522	3.6320
300	0.1588	0.0331	0.1804	16.9572	0.3985	0.1818	0.4248	4.1179
400	0.1257	0.0293	0.1116	15.8453	0.3646	0.1711	0.3340	3.9806
500	0.1055	0.0256	0.0979	11.6312	0.3248	0.1599	0.3129	3.4105

Table 6: **Monte Carlo simulation study results for set II parameters**

(η, μ, a, b)=(0.6, 0.8, 1.6, 1.8)								
	AEs				**ABs**			
n	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$
100	0.7439	0.9728	1.7378	3.7846	0.1439	0.1728	0.1378	1.9846
200	0.6616	0.9150	1.5828	3.4882	0.0616	0.1150	-0.0172	1.6881
300	0.6245	0.8894	1.5460	3.4301	0.0245	0.0894	-0.0540	1.6301
400	0.6305	0.8544	1.5500	3.3321	0.0305	0.0544	-0.0500	1.5321
500	0.6486	0.8357	1.5689	3.2129	0.0486	0.0357	-0.0311	1.4129
	MSEs				**RMSEs**			
n	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{a}$	$\hat{b}$
100	0.3750	0.4610	0.8031	80.4086	0.6124	0.6789	0.8962	8.9671
200	0.2361	0.3399	0.2404	32.2097	0.4859	0.5830	0.4903	5.6754
300	0.1663	0.3075	0.1665	27.1264	0.4078	0.5545	0.4081	5.2082
400	0.1481	0.2735	0.1178	18.0321	0.3849	0.5230	0.3432	4.2464
500	0.1451	0.2350	0.0974	17.9782	0.3809	0.4848	0.3121	4.2401

Concluding comments on the simulation results

(a) The values of parameter estimates converge generally towards the true values with increasing sample size.

(b) The AB, MSE, and RMSE of the parameters decrease generally as the sample size increases.

(b) It is observable that the method of parameter estimation using the maximum likelihood method works well because each estimator performs fairly well and produces low AB, MSE, and RMSE values.

V. APPLICATION

This section of the paper examines the relevance and adaptability of the Kw-GAPR distribution through an examination of HIV/AIDS data. The data set is used to examine the fits of the Kw-GAPR distribution and its sub-models using the goodness-of-fit test and information criterion approaches.

Data set: Wales HIV/AIDS annual death cases data.

The data set consists of 30 HIV/AIDS annual death cases in Wales ranging from 1990 to 2019, which are shown in **Table 7**. The Wales HIV/AIDS annual death cases data was obtained from **https://ourworldindata.org/hiv-aids**.

Table 7: **Wales HIV/AIDS annual death cases data.**

18	22	22	28	32	32	25	18	12	12
12	13	14	15	14	15	16	16	15	16
16	14	14	14	12	15	14	13	12	12

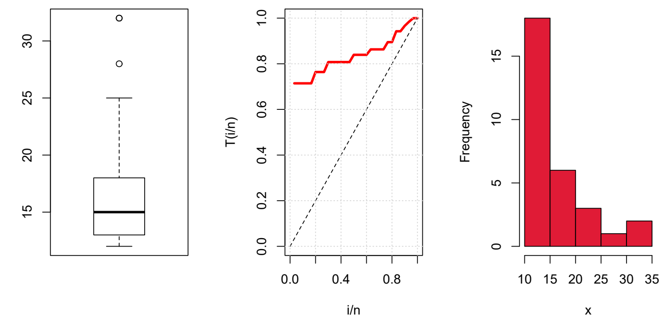
A summary of the most important descriptive statistics of Wales's HIV/AIDS annual death cases data is shown in **Table 8**. From **Table 8**, it is observed that the value of the skewness is positive which implies that the data is right-skewed, bimodal, and the value of the kurtosis is greater than 3 which implies that the data is leptokurtosis (i.e. greater than the value of a normal distribution).

Table 8: **Descriptive statistics for the Wales HIV/AIDS data set.**

Statistic	min.	max.	mean	median	mode	var.	sd.	skewness	kurtosis
Value	12	32	16.8	15	12&14	32.46092	5.697449	1.598322	4.523297

From **Fig. 5**, it is observed that the data has two outliers and the TTT-transform plot depicts that the data has a monotonic increasing (or concave) hazard rate shape defined by [22].

The estimates for the parameters together with their standard errors in parentheses of the fitted models are shown in **Table 9**. Going by the Standard error test which states that a parameter is considered to be significant at 5% level of significance if the standard error is less than half the parameter value, then conclusion can be made that all the parameters of the fitted models are significant at 5% significance level.



(a) (b) (c)

Fig. 5: **(a) Boxplot, (b) TTT-transform plot, and (c) Histogram for the data set**

Table 9: **Summary of the estimates and SEs (in parentheses) for the HIV/AIDS data**

$Model$	$\hat{a}$	$\hat{b}$	$\hat{\eta}$	$\hat{\mu}$
Kw-GAPR	10.834(5.401)	0.263(0.089)	12.864(6.154)	0.020(0.005)
EGAPR	2.920(0.926)	-	0.932(0.183)	0.006(0.001)
GAPR	-	-	1.552(0.594)	0.002(0.0003)
KR	1.588(0.322)	2.813(0.977)	-	0.002(0.0002)
ER	0.007(0.001)	-	-	4.214(1.494)
R	-	-	-	0.003(0.001)

From **Tables 10** and **11**, it is observed that the proposed Kw-GAPR distribution has the smallest values of W^*, A^*, $K-S$, negative log-likelihood, and the information criteria statistics compare to its sub-models. This demonstrates that the Kw-GAPR distribution fits the Wales's HIV/AIDS annual death case data better than its sub-models, though the sub-models also fit the data with the exception of the exponentiated Rayleigh and Rayleigh distribution, which their $p-values < 0.05$.

Table 10: **Summary of goodness-of-fit results for the HIV/AIDS data.**

$Model$	W^*	A^*	$K-S$	$p-value$
Kw-GAPR(proposed)	**0.2931**	**1.6984**	**0.2564**	**0.3875**
EGAPR	0.4119	2.3124	0.2559	0.2928
GAPR	0.4079	2.2920	0.3572	0.0712
KR	0.4574	2.5469	0.2890	0.3133
ER	0.3916	2.2077	0.2117	0.0238
R	0.4454	2.4855	0.3683	0.0006

Fig. 6 depicts a fitted density plot for the Kw-GAPR distribution and its sub-models using Wales HIV/AIDS annual death cases data, and it is observed that the Kw-

Table 11: **The $-\log(\ell)$ and information criteria results for the HIV/AIDS data**

Model	$-log(\ell)$	*AIC*	*BIC*	*CAIC*	*HQIC*
Kw-GAPR(proposed)	**86.8904**	**181.7819**	**187.3867**	**183.3819**	**183.5749**
EGAPR	90.7981	187.5963	191.7999	188.5194	188.9411
GAPR	98.6212	203.2423	206.1388	203.6868	204.1388
KR	93.6623	193.3247	197.5282	194.2477	194.6694
ER	94.2140	200.9598	203.7622	201.4043	201.8563
R	99.3020	198.6039	200.0051	198.7468	199.0522

GAPR distribution best mimics the data, hence exhibiting a promising fit over its sub-models.

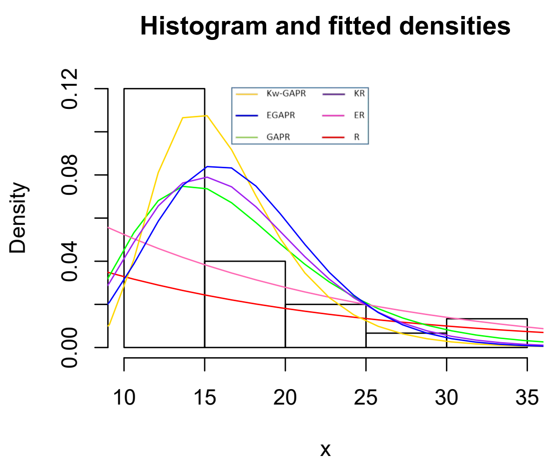


Fig. 6: **The fitted densities for Wales HIV/AIDS annual death cases data**

Concluding comments on the data application

(a) For the Wales HIV/AIDS data set, one can draw a conclusion that, when compared to other models (i.e. sub-models), the Kw-GAPR provides the lowest values for the information criteria, $-log(\ell)$, $K-S$, the W^*, and A^*, as well as the highest p value.

(b) The best-fitting model for the Wales HIV/AIDS data set, from Fig. 6 was Kw-GAPR distribution.

(c) The Rayleigh (R) and Exponentiated Rayleigh (ER) distributions offer a poor fitting for the Wales HIV/AIDS data set, as seen in Table 10.

(d) From the data application results, the Kw-GAPR distribution provides the best fitting among its sub-models, which indicates that the Kw-GAPR distribution has a bigger advantage in fitting this type of data set.

VI. CONCLUSION

The Rayleigh distribution has widely been used as a baseline distribution by mathematicians and statisticians to modify or extend some families of distributions by making the family more flexible. In this paper, a new four-parameterize distribution, namely the Kumaraswamy-Gull Alpha Power Rayleigh distribution, abbreviated as Kw-GAPR distribution has been proposed. Some statistical properties of the proposed Kw-GAPR distribution are investigated such as hazard rate function, quantile function, moments, entropy, and order statistic. The Kw-GAPR parameters' maximum likelihood estimators are determined, and a Monte Carlo simulation study was carried out. The Average Estimates, the Average Biases, the Mean Square Errors, and the Root Mean Square Errors were computed. From the results of the simulation study, the maximum likelihood estimates converge generally to the true value of the parameter as the sample size increases, and there is a decrease (in general) in the Average Biases, the Mean Square Errors, and the Root Mean Square Errors with an increase in the sample size. Finally, the Kw-GAPR distribution was applied to HIV/AIDS data set, and the results demonstrate that, when compared Kw-GAPR to its sub-models, the Kw-GAPR distribution provides the greatest fit for the data set.

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