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TAKE OFF AND LANDING PERFORMANCE OF AN AIRCRAFT UNDER PARABOBOLIC DRAG AND RUNWAY FRICTION

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ABSTRACT

Take-off and landing maneuvers are studied replacing the velocity-square law of atmospheric drag by parabolic polar drag and as such the equations of motion are set up and mostly solved in close form. Hence the distance traveled and Velocity gained on the runway are determined. Variation of the aircraft weight due to fuel consumption, however small may be during travel on the runway, is taken into consideration. A case wherein a variable thrust is exerted leading to movement of the aircraft on the runway with a uniform velocity for some time is also investigated .Finally the fuel consumption on the runway is computed.

INTRODUCTION

Angelo Miele[2] discussed take-off and landing performance, considering conventional velocity-square-law of drag but neglecting variation of the aircraft weight due to fuel consumption on the runway. Kettle,D.[3].,estimated take-off and landing Airborne Paths. SN Maitra[1], present author, incorporated performance of take- off and landing obeying velocity –square law of drag.The equations of motion[1] of the aircraft during airborne phase vis-à-vis for its travel on the runway are written[1] as

$\dot{x} = v$

$T - D_P - \mu R - \frac{W}{g}\dot{v} = 0$ (1)

$R + L - W = 0$

$\dot{W} + cT = 0$ (2)

$L = \frac{1}{2} C_L \rho S v^2$

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$D_P = \frac{1}{2} C_{DO} \rho S v^2 + \frac{k_1 W^2}{v^2}$ (3)

where k_1 =constant, R the normal reaction of the runway, μ the coefficient of runway friction, v the velocity and x the distance, C_L the coefficient of lift, ρ the air density, S the reference area of the aircraft, C_{DO} xero-lift coefficient of drag, c the specific fuel consumption, D_P the parabolic polar drag, W the instantaneous weight,W_0 the initial weight of the aircraft, ie at the time of beginning of its movement on the runway, T the thrust,

the dot sign a derivative with respect to time t and g the acceleration due to gravity.Since we have considered variation of the weight of the aircraft due to its fuel consumption on the runway, another equation (2) arises in this feature unlike Miele's[1] observation.

MOTION OF THE AIRCRAFT ON THE RUNWAY WITH CONSTANT THRUST

The initial and boundary conditions are prescribed as

At time t=0, $W=W_0$, v=0, x=0 for take-off (4)

while the distance traveled by the aircraft in tme t_k is x_k whereas

At time t=0, $W=W_L$, $v=v_L$, x=0 for landing (5)

where the distance covered in coming to rest in time t_L of landing is x_L .Since the fuel consumption for travel on the runway is very small compared to the fuel intake, we rule out choosing the instantaneous weight as the independent variable for solutions to the governing equations. Then solution to equation (2) with (4) in consequence of constant thrust T yields the instantaneous weight

$$W=W_0-cTt \qquad (6)$$

Combining equations (1) ,(2) (3) and (6) one gets,

$$\frac{dt}{dv}=\frac{k(b-t)}{2a^2-v^2-\frac{a_1^2}{v^2}} \qquad (7)$$

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$$\text{where } a^2=\frac{T-\mu W_0}{\rho S(C_{Do}-\mu C_L)},\ a_1^2=\frac{2k_1W_0^2}{\rho S(C_{Do}-\mu C_L)},\ k=\frac{2cT}{\rho Sg(C_{Do}-\mu C_L)},\ b=\frac{W_0}{cT} \qquad (8)$$

$$\text{Or, } \frac{1}{k(b-t)}\frac{dt}{dv}=\frac{v^2}{B^4-(v^2-a^2)^2}=\frac{1}{2B^2}\left(\frac{B^2+a^2}{B^2+a^2-v^2}-\frac{B^2-a^2}{B^2-a^2+v^2}\right) \qquad (9.1)$$

For minor approximation in the expression of parabolic drag, W is replaced by its initial value W_0 on account of small variation of W on the runway. (9.1) leads to

$$\frac{1}{(b-t)}\frac{dt}{dv}=\frac{q_1}{p_1^2-v^2}-\frac{q_2}{p_2^2+v^2} \qquad (9.2)$$

$$\text{where } B^4=a^4-a_1^2,\ q_1=\frac{k(B^2+a^2)}{2B^2}\quad q_2=\frac{k(B^2-a^2)}{2B^2} \qquad (10)$$

$$p_1^2=B^2+a^2,\ p_2^2=B^2-a^2 \qquad (11)$$

Take –off performance :

With relevant initial conditions (4) solution to equation (9) turns out to be

$$-\log\left(1-\frac{t}{b}\right)=\frac{q_1}{2p_1}\log\frac{p_1+v}{p_1-v}-\frac{q_2}{p_2}\tan^{-1v/p_2}$$

$$\log\left(1-\frac{t}{b}\right)=\frac{q_2}{p_2}\tan^{-1v/p_2}-\frac{q_1}{2p_1}\log\frac{p_1+v}{p_1-v}$$

In other words taking $n_1=\frac{q_1}{2p_1}$, $n_2=\frac{q_2}{p_2}$ and simplifying for $t=t_k$ and $v=v_k$ we get

$$t_k= b\{1-(\frac{p_1-v_k}{p_1+v_k})^{n_1}e^{n_2\tan^{-1v_k/p_2}}\} \qquad (12)$$

which gives the time taken to take off with lift-off velocity v_{k}..

In order to find the take-off run x_k we rewrite equation(7) as

$$\frac{dx}{dv}=\frac{k(b-t)v}{2a^2-v^2-\frac{a_1^2}{v^2}} ; \qquad t= b\{1-(\frac{p_1-v}{p_1+v})^{n_1}e^{n_2\tan^{-1v/p_2}}\}$$

Eliminating t between these two equations and recalling (9),we get

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$$\frac{dx}{dv}=b\left(\frac{p_1-v}{p_1+v}\right)^{n_1}e^{n_2tan^{-1v/p_2}}(\frac{q_1}{p_1^2-v^2}-\frac{q_2}{p_2^2+v^2})v \qquad (13)$$

Because of initial conditions (4),integral of (13) becomes

$$x_k=\int_0^{v_k}\left(\frac{p_1-v}{p_1+v}\right)^{n_1}e^{n_2tan^{-1v/p_2}}(\frac{q_1}{p_1^2-v^2}-\frac{q_2}{p_2^2+v^2})vdv \qquad (14)$$

which can be evaluated numerically with given numerical values of the parameters .

Landing performance :

In view of (7) for landing performance, the equation of motion with reversed constant thrust is modified as

$$\frac{dt}{dv}=\frac{-k(b-t)}{2a^2+v^2+-\frac{a_1^2}{v^2}} \qquad (15)$$

which with the same technique as earlier is reduced to the form

$$\frac{1}{k(b-t)}\frac{dt}{dv}=\frac{-v^2}{(v^2+a^2)^2-D^4}=-\frac{1}{2D^2}(\frac{a^2+D^2}{v^2+a^2+D^2}-\frac{a^2-D^2}{v^2+a^2-D^2})$$

Or, $-\frac{1}{(b-t)}\frac{dt}{dv} = \frac{k}{2D^2}\left(\frac{a^2+D^2}{v^2+a^2+D^2} - \frac{a^2-D^2}{v^2+a^2-D^2}\right)$; $D^4 = a^4 - a_1^2$ (16)

which subject to the initial conditions(4) admits of the solution:

$$\mathrm{Log}\left(1-\frac{t}{b}\right) = \frac{k}{2D^2}\left[\left\{r_1 \tan^{-1}\frac{v}{r_1} - r_2 \tan^{-1}\frac{v}{r_2}\right\} - \left\{r_1 \tan^{-1}\frac{V_L}{r_1} - r_2 \tan^{-1}\frac{V_L}{r_2}\right\}\right] \quad (17)$$

$$t = b\left[1 - D_L e^{\frac{k}{2D^2}\left\{r_1 \tan^{-1}\frac{v}{r_1} - r_2 \tan^{-1}\frac{v}{r_2}\right\}}\right] \quad (18)$$

where $r_1^2 = a^2 + D^2$ and $r_2^2 = a^2 - D^2$

$$D_L = e^{\frac{-k}{2D^2}\left\{r_1 \tan^{-1}\frac{V_L}{r_1} - r_2 \tan^{-1}\frac{V_L}{r_2}\right\}} \quad (19)$$

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Putting v=-0 in (18) and using (19) is found the time t_L of run during landing the aircraft :

$$t_L = b\left\{1 - e^{\left(\frac{-k}{2D^2} r_1 \tan^{-1}\frac{V_L}{r_1} - r_2 \tan^{-1}\frac{V_L}{r_2}\right)}\right\} \quad (20)$$

To find the distance x_L traveled on the runway while landing we multiply (15) by $\frac{dx}{dt} = v$ and substitute for (b-t) from (18)

$$\frac{dx}{dv} = \frac{-kbD_L v^3 e^{\frac{k}{2D^2}\left\{r_1 \tan^{-1}\frac{v}{r_1} - r_2 \tan^{-1}\frac{v}{r_2}\right\}}}{2v^2a^2 + v^4 + a_1^2} \quad (21)$$

$$x_L = kbD_L \int_0^{V_L} v^3 f(v)dv \quad (22)$$

where f(v) is defined as above and this integral has to be evaluated by numerical process with given numerical values of the parameters. Needless to mention that because of constant thrust, there is uniform rate of fuel consumption.

OPTIMUM VALUE OF PARABOLIC DRAG

Differentiating D_P from (3) with respect to v^2 and equating to zero,

$$\frac{dD_P}{dv^2} = \frac{1}{2}C_{D0}\rho S - k_1 W^2 \frac{1}{v^4} = 0, \quad \frac{d^2D_p}{d(v^2)^2} = k_1 W^2 \frac{2}{v^6} > 0 \quad (23)$$

which implies that the parabolic drag is minimum when the velocity is

$$v = \left(2\frac{k_1 W^2}{C_{Do}\rho S}\right)^{1/4} \quad (24)$$

Owing to very small consumption of the fuel during the run on the runway, variation of the weight W of the aircraft is somehow neglected and as such differentiation of W with respect to v^2 does not take place in equation (23). Substituting for v from (24) in (3), we get

$$D_P=\frac{1}{2}C_{Do}\rho S(2\frac{k_1W^2}{C_D\rho S})^{1/2}+k_1W^2(\frac{C_{Do}\rho S}{2k_1W^2})^{1/2}=2W\sqrt{C_{Do}\rho S\frac{k_1}{2}} \quad (25)$$

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Replacing v_k in (12) and (14) respectively by expression for v given by (24),we obtain the velocity acquired and distance covered corresponding to the minimum parabolic drag (25) during the take-off run. Similarly we can find out the relevant velocity and distance for landing run and the times reckoned for these two runs can also be determined.

UNIFORM VELOCITY OF THE AIRCRAFT ON THE RUNWAY UNDER PARABOLIC DRAG DURING TAKE OFF /LANDING

Uniform velocity for some time during take-off : To execute a uniform velocity on the runway for some time, a constant thrust is applied so that the aircraft is accelerated to a certain velocity overcoming the resistive force comprising the drag and runway friction. Thereafter it moves with this velocity as uniform velocity applying a controlled variable thrust equated to the resistive force as above. Hence recalling the preceding equations (1) to (3) the instantaneous thrust is regulated as

$$T= D_P+\mu R = \frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+\frac{k_1}{v^2}W^2+\mu W \quad (26)$$

which is substituted in equation (6) to obtain the fuel consumption :

$$\frac{dW}{dt}=-c\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+\frac{k_1}{v^2}W^2+\mu W\} \quad (27)$$

To solve this differential equation we need to adapt some procedure of approximation but that too without sacrifice of accuracy and generality. We replace W^2 by WW_0 in (27) because of small variation of the weight of the aircraft due to insignificant fuel consumption on the runway. Hence the equation reduces to the form

$$\frac{dW}{dt}=-c\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+W(\frac{k_1}{v^2}W_0+\mu)\} \quad (28)$$

Integrating (28) subject to the initial conditions (4),one gets time t in terms of weight W

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$$ct= \frac{1}{(\frac{k_1}{v^2}W_0+\mu)}\log\frac{\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+W_0(\frac{k_1}{v^2}W_0+\mu)\}}{\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+W(\frac{k_1}{v^2}W_0+\mu)\}} \quad (29)$$

Or, $\frac{\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+W(\frac{k_1}{v^2}W_0+\mu)\}}{\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2+W_0(\frac{k_1}{v^2}W_0+\mu)\}} = e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}$ $\quad W_0 > W$

Or, $\frac{1}{2}\rho S(C_o - \mu C_L)\, v^2 + W(\frac{k_1}{v^2}W_0 + \mu)$

$= \{\frac{1}{2}\rho S(C_{Do} - \mu C_L)\, v^2 + W_0(\frac{k_1}{v^2}W_0 + \mu)\}\, e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}$

Or, $W(\frac{k_1}{v^2}W_0 + \mu)$

$= W_0(\frac{k_1}{v^2}W_0 + \mu)\, e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct} - \{\frac{1}{2}\rho S(C_{Do} - \mu C_L)\, v^2\}\{1 - e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}\}$

$$W=\frac{W_0(\frac{k_1}{v^2}W_0+\mu)\, e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}-\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2\}\{1-e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}\}}{(\frac{k_1}{v^2}W_0+\mu)} \qquad (30)$$

gives the weight of the aircraft at time t on the run way due to fuel consumption ∈.However, the fuel consumption ∈ can be expressed in a proper form:

$$\in= W_0 - W = \frac{\{W_0\left(\frac{k_1}{v^2}W_0+\mu\right)+\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2\}\{1-e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}\}}{(\frac{k_1}{v^2}W_0+\mu)} \qquad (30.1)$$

which shows that greater the chosen velocity , more is the fuel consumption. Similarly in case of landing, the aircraft can be driven with touchdown velocity or with a uniform velocity for some time on the runway maneuvering the controlled variable thrust and so the fuel consumption during that time can be computed as above. If the chosen uniform velocity is the velocity that ensures minimum parabolic drag,it facilitates faster movement of the aircraft on the runway.

Maintenance of this controlled thrust T for this feature is done using (26) and (30):

$$T = \frac{1}{2}\rho S(C_{Do} - \mu C_L)\, v^2+\mu\frac{W_0(\frac{k_1}{v^2}W_0+\mu)\, e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}-\{\frac{1}{2}\rho S(C_{Do}-\mu C_L)\, v^2\}\{1-e^{-\left(\frac{k_1}{v^2}W_0+\mu\right)ct}\}}{(\frac{k_1}{v^2}W_0+\mu)} \qquad (31)$$

which suggests that greater the thrust, greater is the velocity.

UNIFORM VELOCITY ON THE RUNWAY WHILE OVERCOMING CONVENTIONAL DRAG AND RUNWAY FRICTION

In this case equation (26) and (27) are amended by removing the term containing k_1 so that

$T=D+\mu R$

$T=\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W$ (32)

$\frac{dW}{dt}=-c\{\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W\}$ (33)

Solving (32) and applying the initial conditions (4) is obtained

$-c\mu t=\log\frac{\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W}{\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W_0}$

Or, $t=\frac{1}{c\mu}\log\frac{\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W_0}{\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W}$ (34)

Or, $\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W=(\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W_0)e^{-c\mu t}$

Hence the instantaneous weight of the aircraft with this uniform velocity is given by

$W=(\frac{1}{2}\rho S(C_D-\mu C_L)\,\frac{v^2}{\mu}+W_0)e^{-c\mu t}-\frac{1}{2}\rho S(C_D-\mu C_L)\,\frac{v^2}{\mu}$

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Or, $W=W_0e^{-c\mu t}-\frac{1}{2}\rho S(C_D-\mu C_L)\,\frac{v^2}{\mu}(1-e^{-c\mu t})$ (35)

Hence the fuel expenditure for movement of the aircraft for some time with the chosen uniform velocity v overcoming the drag obeying velocity-square law and runway friction during take-off or landing becomes

$\beta=W_0-W=\{\frac{1}{2}\rho S(C_D-\mu C_L)\frac{v^2}{\mu}+W_0\}(1-e^{-c\mu t})$ (36)

Maintenance of the thrust for this design of uniform velocity on the runway is given by virtue of (32and (35):

$T=\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2+\mu W_0e^{-c\mu t}-\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2(1-e^{-c\mu t})$

Or, $T=\{\mu W_0+\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2\}e^{-c\mu t}$ (37)

The results can be verified in the foregoing analysis by putting $k_1=0$ entailing parabolic drag turning into square-law- velocity drag. Equation (36) reveals that the thrust increases with the increase in velocity and the maneuvering thrust is a reducing exponential function of time from its initial value T_0 :

$T_0=\mu W_0+\frac{1}{2}\rho S(C_D-\mu C_L)\,v^2$ (38)

On scrutinizing the relevant thrust expressed by equation (31),it plays the same role as above, viz, equation (37).

CONCLUSION

In this paper the fuel consumption is only partially neglected during its travel on the runway in determining all the parameters, viz, thrust, parabolic drag, weight, fuel consumption, velocity acquired, distance traveled, etc.However, equation (27) can be solved without resorting to any technique of approximation and subtlety and hence is rewritten as

$$\frac{dW}{dt} = -c\{\frac{1}{2}\rho S(C_{Do} - \mu C_L)\, v^2 + \frac{k_1}{v^2} W^2 + \mu W\}$$

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$$= -c\frac{k_1}{v^2}\{\frac{v^4}{2k_1}\rho S(C_{Do} - \mu C_L) + W^2 + 2\mu\frac{v^2}{2k_1}W\}$$

$$= -c\frac{k_1}{v^2}\{\left(W + \mu\frac{v^2}{2k_1}\right)^2 - \left(\mu\frac{v^2}{2k_1}\right)^2 + \frac{v^4}{2k_1}\rho S(C_{Do} - \mu C_L)\}$$

Letting $\left(\mu\frac{v^2}{2k_1}\right)^2 - \frac{v^4}{2k_1}\rho S(C_{Do} - \mu C_L) = E^2$ (39)

Is obtained the rate of fuel consumption as

$$\frac{dW}{dt} = -c\frac{k_1}{v^2}\{\left(W + \mu\frac{v^2}{2k_1}\right)^2 - E^2\} \quad (40)$$

which is solved using the initial conditions (4):

$$= \frac{1}{2E}\log\{\left(\frac{W+\mu\frac{v^2}{2k_1}-E}{W+\mu\frac{v^2}{2k_1}+E}\right)\left(\frac{W_0+\mu\frac{v^2}{2k_1}+E\}}{W_0+\mu\frac{v^2}{2k_1}-E}\right)\} = -c\frac{k_1}{v^2}t$$

$$\frac{W+\mu\frac{v^2}{2k_1}-E}{W+\mu\frac{v^2}{2k_1}+E} = \left(\frac{W_0+\mu\frac{v^2}{2k_1}-E\}}{W_0+\mu\frac{v^2}{2k_1}+E}\right) e^{-2Ec\frac{k_1}{v^2}t}$$

Or, $$\left(\frac{w+\mu\frac{v^2}{2k_1}}{E}\right) = \frac{(W_0+\mu\frac{v^2}{2k_1}+E)+(W_0+\mu\frac{v^2}{2k_1}-E\})e^{-2Ec\frac{k_1}{v^2}t}}{(W_0+\mu\frac{v^2}{2k_1}+E)-(W_0+\mu\frac{v^2}{2k_1}-E\})e^{-2Ec\frac{k_1}{v^2}t}}$$

Or, $$W = E\,\frac{(W_0+\mu\frac{v^2}{2k_1}+E)+(W_0+\mu\frac{v^2}{2k_1}-E\})e^{-2Ec\frac{k_1}{v^2}t}}{(W_0+\mu\frac{v^2}{2k_1}+E)-(W_0+\mu\frac{v^2}{2k_1}-E\})e^{-2Ec\frac{k_1}{v^2}t}} - \mu\frac{v^2}{2k_1} \quad (41)$$

represents the weight of the aircraft at time t. Hence the fuel consumption α :

$$\alpha = W_0 - W = W_0 - \; E\,\frac{(W_0+\mu\frac{v^2}{2k_1}+E)+(W_0+\mu\frac{v^2}{2k_1}-E)e^{-2Ec\frac{k_1}{v^2}t}}{(W_0+\mu\frac{v^2}{2k_1}+E)-(W_0+\mu\frac{v^2}{2k_1}-E\})e^{-2Ec\frac{k_1}{v^2}t}} + \mu\frac{v^2}{2k_1} \qquad (42)$$

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