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An Inventory Model for Decaying Items with Life Time Under Time Dependent Storage Cost And Price Dependent Demand

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Abstract

In the proposed study we shall developed and analyze inventory models with variable holding cost with other different realistic business environment. More specifically, we considered this model with life time assumption of the products since practically more products in the market have their life time. Deterioration of the products is considered only after the life time of the products. Demand will be any twice differential function of price. It is assumed that selling price is constant with in inventory cycle and demand can be backlogged.

Keywords: Inventory, shortage, demand, retrogistics.

1. Introduction

For all businesses, inventory management is a crucial role. The inventory management department's major goal is to keep track of materials from purchase to sale, making judgments about how much and when to acquire certain things to minimise excess or unplanned stock outs.

Manna and Chaudhuri (2004) created an infinite time-horizon deterministic economic order quantity inventory model with deterioration based on a discounted cash flows method with a non-linear demand rate over time. Under a trade-credit strategy, the impact of inflation and the time value of money were also considered. Manna et al. proposed a replenishment approach for EOQ models with time-dependent quadratic demand and shortfalls (2007). Patra et al. (2010) investigated a non-linear order-level EOQ model for degrading items in a single warehouse system with price-dependent demand. The economic production quantity model for degrading commodities with non-linear holding costs under inflationary conditions was studied by Valliathal and Uthayakumar (2011). Pando et al. (2013) created an inventory model with a non-linear holding cost that is dependent on time and quantity. Trivedi et al. (2014) looked into a deterministic inventory model with time-dependent quadratic demand and non-linear holding costs.

Barik et al. (2014) looked at the inventory system for perishable commodities, taking into account the time proportional deterioration rate. The EOQ was established in order to reduce the average total cost per unit of time. With a finite time horizon, a time dependent demand rate was applied. The non-linear holding cost with shortage was taken into account. Ray (2014) established an economic order quantity model for a constantly deteriorating item for which the supplier allows a fixed payment delay. Jose et al. (2015) investigated a partial backordering and non-linear unit holding cost economic order quantity inventory model. For degrading objects, Tripathi et al. (2016) provided an economic order quantity model. The demand rate and holding cost in this model were non-linear functions. San-Jose et al. (2017) developed a non-linear holding cost, partial backlog, and ramp-type demand economic order quantity model. Khalilpourazari et al. proposed a multi-item multi-constrained economic order quantity model with non-linear unit holding cost and partial backordering (2017).

Tripathi developed a deterministic inventory model with non-linear time-dependent and stock-dependent holding costs for non-increasing time-sensitive demand (2018). Pando et al. (2019) investigated a deterministic inventory model with a stock-dependent demand pattern and a cumulative holding cost that was a non-linear function of both time and stock level. Manna created an economic order quantity model for decaying goods with non-linear demand, inflation, temporal discounting, and a trade credit policy (2019). Barron et al. (2020) proposed a non-linear stock-dependent demand and non-linear holding cost EOQ inventory model. This inventory model was created from the perspective of a retailer who receives a trade credit period from a supplier.

The EOQ is well-known for determining the best order quantity to reduce overall inventory expenses. Despite the fact that this inventory model lays the foundation for inventory systems, it is based on reasonable assumptions.

2. Notation and assumption

1) $I(t)$ be the inventory level at time t, $t \geq 0$.
2) Now t_1 is the time at which shortage starts and T is the length of the replenishment cycle, $0 \leq t_1 \leq T$.
3) Replenishment rate is infinite and lead time is zero.
4) There is no repair or replenishment of deteriorated units during the period.
5) A single item is considered over the prescribed period T units of time.
6) S is the initial inventory level after fulfilling backorders.

7) μ is the life time of items and deterioration of the items is considered only after the life time of items and θ is the constant deterioration rate.

8) Demand is represented by a general function. i. e. the demand function can be any twice differentiable function of price .Following assumption is made concerning the demand function.

1) $D' = \frac{d\,D(p)}{d\,p} < 0$ for all $p \in (0,\infty)$

2) The marginal revenue $\frac{d\{pD(p)\}}{d\,D} = p + \frac{D(p)}{D'}$ is a strictly increasing function of p and thus $\frac{1}{D(p)}$ is a convex function of p.

9) Holding cost per unit per time C_1 (t) will be linear function of time as (a + b t) i.e. holding cost will be variable with respect to time. Where 'a' & 'b' are positive constants such that $0 < a < b \leq 1$ and $b \neq 0$.

10) Unsatisfied demand is backlogging at a rate exp $(-\delta\, t)$ where t is the time up to next replenishment. The backlogging parameter δ is a positive constant.

11) C_2, C_3, & C_4 are the unit purchase cost per unit, shortage cost per unit per unit time and unit cost of lost sale respectively.

12) C' is the inventory ordering cost per order.

3. Formulation & Solution of the model

During the period $(0, \mu)$ the inventory level gradually diminishes due to market demand only. After life time deterioration can take place, therefore during the period (μ, t_1) the inventory level decreases due to the market demand and deterioration of items and falls to zero at time t_1. The period (t_1, T) is the period of shortage which is partially backlogged .The depletion of inventory given in the figure (1).

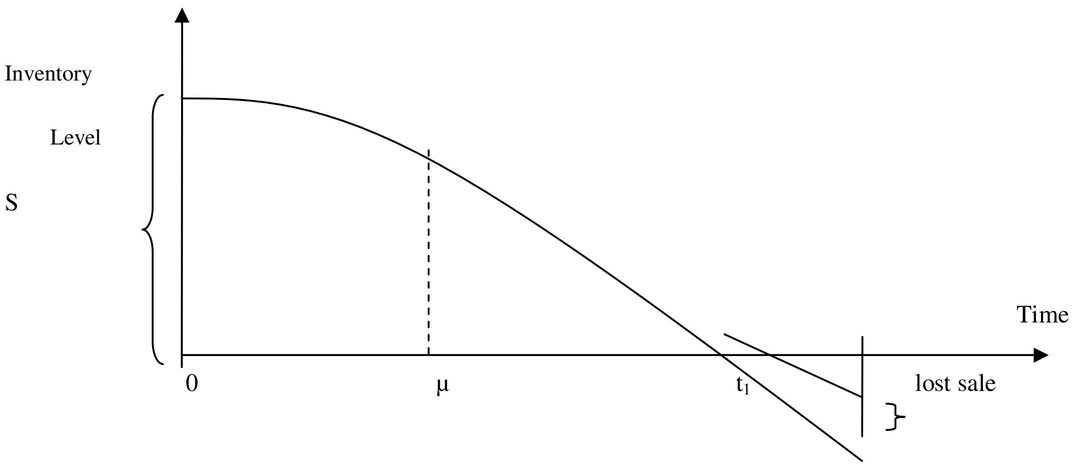

Figure 1 : Inventory time graph with shortage

The governing differential equation of the proposed inventory system in the interval (0, T) are

$$I'(t) = -D(p) \qquad 0 \le t \le \mu \qquad \ldots\ldots (1)$$

$$I'(t) + \theta I(t) = -D(p) \qquad \mu \le t \le t_1 \ldots\ldots (2)$$

$$I'(t) = -D(p)\, e^{-\delta t} \qquad t_1 \le t < T \ldots\ldots. (3)$$

With boundary conditions

$$I(0) = S,\ I(t_1) = 0 \qquad \ldots\ldots. (4)$$

Solution of the equations (1) & (2) are

$$I(t) = S - t\,D(p) \qquad 0 \le t \le \mu \ldots\ldots.. (5)$$

$$I(t) = \frac{D(p)}{\theta}\left\{e^{\theta(t_1 - t)} - 1\right\} \quad \mu \le t \le t_1 \ldots\ldots\ldots (6)$$

By the equation (5) & (6), we get

$$S = \frac{D(p)}{\theta}\left\{\theta t + e^{\theta(t_1 - t)} - 1\right\}$$

Putting $t = \mu$

$$S = \frac{D(p)}{\theta}\left\{\theta \mu + e^{\theta(t_1 - \mu)} - 1\right\} \ldots\ldots\ldots\ldots (7)$$

From equations (5) & (7)

$$I(t) = \frac{D(p)}{\theta}\{\theta\,\mu + e^{\theta(t_1-\mu)} - 1\} - t\,D(p)$$

$$= D(p)(\mu - t) + \frac{D(p)}{\theta}\{e^{\theta(t_1-\mu)} - 1\} \qquad 0 \le t \le \mu \ \ldots\ldots\ldots (8)$$

Now, solution of the equation (3)

$$I(t) = \frac{D(p)}{\delta}(e^{-\delta t} - e^{-\delta t_1}) \qquad t_1 \le t < T \ldots\ldots (9)$$

Hence inventory holding cost (C_H) during the period (0, T) becomes

$$C_H = (a + b\,t)\left[\int_0^{\mu} I(t)\,dt + \int_{\mu}^{t_1} I(t)\,dt\right]$$

$$= \frac{a\,D(p)}{2\theta^2}\{2(\theta\mu + 1)e^{\theta(t_1-\mu)} + \mu^2\theta^2 - 2\theta t_1 - 2\} + \frac{D(p)\,b}{6\,\theta^3}$$

$$[\theta^3\mu^3 - 3\,\theta^2 t_1^2 - 6\,t_1\theta + 6e^{\theta(t_1-\mu)}\{\theta\mu(\theta + 1) + 1\} - 6]$$

... (10)

The cost due to deterioration of units (C_D) during the period (0, T) is given by

$$C_D = C_2\left[\int_{\mu}^{t_1} \theta\, I(t)\,dt\right]$$

$$= \frac{D(p)C_2}{\theta}\int_{\mu}^{t_1}\{e^{\theta(t_1-t)} - 1\}\,dt$$

$$= \frac{D(p)\,C_2}{\theta}[e^{\theta(t_1-\mu)} - \theta(t_1 - \mu) - 1] \qquad \ldots\ldots (11)$$

Cost due to shortage of units (C_S) during the given period is given by,

$$C_S = -C_3\int_{t_1}^{T} I(t)\,dt$$

$$= -C_3\int_{t_1}^{T}\frac{D(p)}{\delta}(e^{-\delta t} - e^{-\delta t_1})\,dt$$

$$= \frac{D(p)C_3}{\delta^2}\left[\left(e^{-\delta T} - e^{-\delta t_1}\right) + \delta e^{-\delta t_1}(T - t_1)\right] \qquad \text{..... (12)}$$

Opportunity cost due to lost sales (C_O) is given by

$$C_0 = C_4\int_{t_1}^{T}(1-e^{-\delta t})D(p)dt$$

$$= \frac{D(p)C_4}{\delta}\left\{\left(e^{-\delta T} - e^{-\delta t_1}\right) + \delta(T - t_1)\right\} \qquad \text{...... (13)}$$

The total average cost C (t_1, T|p) of the inventory system per unit time is given by

$$C(t_1, T|p) = \frac{(C' + C_H + C_D + C_S + C_o)}{T} = \frac{R}{T}$$

Where R is the total cost of the inventory cycle

$$C(t_1, T) = \frac{1}{T}\Big[C' + \frac{a\,D(p)}{2\theta^2}\left\{2(\theta\mu + 1)e^{\theta(t_1-\mu)} + \mu^2\theta^2 - 2\theta t_1 - 2\right\} +$$

$$\frac{D(p)b}{6\theta^3}\left[\theta^3\mu^3 - 3\theta^2 t_1^2 - 6t_1\theta + 6e^{\theta(t_1-\mu)}\{\mu\theta(\theta+1)+1\} - 6\right] + \frac{D(p)C_2}{\theta}\left[e^{\theta(t_1-\mu)} - \theta(t_1-\mu) - 1\right]$$

$$+\frac{D(p)C_3}{\delta^2}\left[\left(e^{-\delta T} - e^{-\delta t_1}\right) + \delta e^{-\delta t_1}(T - t_1)\right] + \frac{D(p)C_4}{\delta}\left\{\left(e^{-\delta T} - e^{-\delta t_1}\right) + \delta(T - t_1)\right\}\Big] \qquad \text{...... (14)}$$

Obviously, the above cost function for optimal value of t_1& T can be evaluated by the minimizing C (t_1, T). Before minimizing C (t_1, T), we shall proof some lemmas in this concern, that is theoretically analysis. Some lemmas are following as.

Lemma: 1. $\left[\theta^3\mu^3 + 6\{\theta\mu(\theta+1) + 1\}e^{\theta(t_1-\mu)} - 3\theta^2 t_1^2 - 6t_1\theta - 6\right] > 0$, if $t_1 \geq \mu$

Proof:we shall proof the lemma by contradiction, we get

If given function of variable t_1 will be negative .then we get

Now

$$\left[\theta^3\mu^3 + 6\{\theta\mu(\theta+1) + 1\}e^{\theta(t_1-\mu)} - 3\theta^2 t_1^2 - 6t_1\theta - 6\right] < 0$$

$$\left[\theta^3\mu^3 + 6(\theta^2\mu + \theta\mu + 1)e^{\theta(t_1-\mu)} - 3\theta^2 t_1^2 - 6t_1\theta - 6\right] < 0$$

By $e^{\theta\,(t_1-\mu)} = 1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2$(by neglecting higher terms)

$$\left[\theta^3\mu^3+6\theta^2\mu\{1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+6\theta\mu+6\theta^2\mu(t_1-\mu)\right.$$

$$+3\theta^3\mu(t_1-\ \mu)^2+6+6\theta t_1-6\mu\theta+3\theta^2t_1^2+3\theta^2\mu^2-6t_1\theta^2\mu$$

$$\left.-3\theta^2t_1^2-6t_1\theta-6\right]<0$$

$$\left[\theta^3\mu^3+6\theta^2\mu\{1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+6\theta^2\mu(t_1-\mu)+3\mu\theta^3(t_1-\mu)^2+\right.$$

$$\left.3\theta^2\mu^2-6t_1\theta^2\mu\right]<0$$

$$\left[\theta^3\mu^3+6\theta^2\mu\ +6\theta^2\mu\{\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+6\theta^2\mu t_1-6\theta^2\mu^2+\right.$$

$$\left.3\mu\theta^3(t_1-\mu)^2+3\theta^2\mu^2-6t_1\theta^2\mu\right]<0$$

$\left[\theta^3\mu^3+6\theta^2\mu\{1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+3\mu\theta^3(t_1-\mu)^2-3\theta^2\mu^2\right]<0$ Since $t_1\geq\mu$ so

numerically $\left[\theta^3\mu^3+6\theta^2\mu\{1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+3\mu\theta^3(t_1-\mu)^2\right]>3\theta^2\mu^2$

So $\left[\theta^3\mu^3+6\theta^2\mu\{1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\}+3\mu\theta^3(t_1-\mu)^2-3\theta^2\mu^2\right]>0$

Therefore our assumption is wrong .Hence the proof

Lemma: **2.** If $\left[e^{\theta\,(t_1-\mu)}-\theta(t_1-\mu)-1\right]$ will be positive for $t_1\geq\mu$

Proof: we shall proof the lemma by contradiction, we get

If given function of variable t_1 will be negative .then we get

Now

If $\left[e^{\theta\,(t_1-\mu)}-\theta(t_1-\mu)-1\right]<0$

$$\left[1+\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2-\theta(t_1-\mu)-1\right]<0$$

$[\frac{\theta^2}{2}(t_1-\mu)^2] < 0$

Since $t_1 \geq \mu$so numerically $[\frac{\theta^2}{2}(t_1-\mu)^2] > 0$.Therefore our assumption is wrong .Hence the proof

Lemma: **3.** If $T(\delta T - 2) > t_1(\delta t_1 - 2)$ where δ is the backlogging parameter, always being positive for $T > t_1$. Then $(e^{-\delta T} - e^{-\delta t_1}) > 0$

Proof: If $T(\delta T - 2) > t_1(\delta t_1 - 2)$ for $T > t_1$

Multiplying by $\frac{\delta}{2}$on both side of the above equation

$$\frac{\delta}{2}T(\delta T - 2) > \frac{\delta}{2}t_1(\delta t_1 - 2)$$

$$\frac{\delta^2 T^2}{2} - \delta T > \frac{\delta^2 t_1^2}{2} - \delta t_1$$

Adding one on both side of the above equation

$1 - \delta T + \frac{\delta^2 T^2}{2} > 1 - \delta t_1 + \frac{\delta^2 t_1^2}{2}$.... (15)

Since

$e^{-\delta T} = 1 - \delta T + \frac{\delta^2 T^2}{2}$(by neglecting higher power of δ)

$e^{-\delta t_1} = 1 - \delta t_1 + \frac{\delta^2 t_1^2}{2}$(by neglecting higher power of δ)

From equation, (6.15)

$$e^{-\delta T} > e^{-\delta t_1}$$

Therefore $(e^{-\delta T} - e^{-\delta t_1}) > 0$

Hence the proof

Lemma: **4.** Cost function $C(t_1, T)$ is a positive function with $R > 0$

Proof: Since $T > t_1$, by lemma 1, lemma 2, and lemma 3 we can easily proof that

$R > 0$

Therefore cost function C (t_1, T) will be a positive function with $R > 0$

Hence the proof

Theorem: 1. If $\frac{\partial^2 C}{\partial T^2} > 0$ where C (t_1, T) is a cost function and δ is a backlogging parameter, will be always positive. Then $T(\delta T - 2)(C_3 + \delta C_4) > C_3 t_1 (\delta t_1 - 2)$ iff $(1-e^{-\delta T}) > 0$ that is, $(1-e^{-\delta T})$ will be positive.

Proof: From the double derivation of the cost function, we get

$$\frac{\partial^2 C}{\partial T^2} = \frac{2R}{T^3} - \frac{2D(p)}{T^2}\left[-\frac{C_3}{\delta}(e^{-\delta T} - e^{-\delta t_1}) + C_4(1-e^{-\delta T})\right] +$$

$$\frac{D(p)}{T}\left[C_3 e^{-\delta T} + C_4 \delta e^{-\delta T}\right]$$

Since R is the positive function. Given that $\frac{\partial^2 C}{\partial T^2} > 0$ and $(1-e^{-\delta T})$ will be positive.

Now our problem is to proof only $\left[C_4(1-e^{-\delta T}) - \frac{C_3}{\delta}(e^{-\delta T} - e^{-\delta t_1})\right]$ function will be negative for $\frac{\partial^2 C}{\partial T^2} > 0$

$$\Rightarrow \left[C_4(1-e^{-\delta T}) - \frac{C_3}{\delta}(e^{-\delta T} - e^{-\delta t_1})\right] < 0$$

$$\Rightarrow \frac{C_3}{\delta}(e^{-\delta T} - e^{-\delta t_1}) > C_4(1-e^{-\delta T})$$

$$\Rightarrow C_3 e^{-\delta T} - C_3 e^{-\delta t_1} > \delta C_4 - \delta C_4 e^{-\delta T}$$

$$\Rightarrow C_3 e^{-\delta T} + \delta C_4 e^{-\delta T} > \delta C_4 + C_3 e^{-\delta t_1}$$

$$\Rightarrow e^{-\delta T}(C_3 + \delta C_4) > \delta C_4 + C_3 e^{-\delta t_1} \qquad \ldots..(16)$$

Now

$e^{-\delta T} = 1-\delta T + \frac{\delta^2 T^2}{2}$(by neglecting higher power of δ)

$e^{-\delta t_1} = 1-\delta t_1 + \frac{\delta^2 t_1^2}{2}$(by neglecting higher power of δ)

From inequality (16)

$(1-\delta T + \frac{\delta^2 T^2}{2})\ (C_3 + \delta C_4) > \delta C_4 + C_3\ (1-\delta t_1 + \frac{\delta^2 t_1^2}{2})$

$\Rightarrow (-\delta T + \frac{\delta^2 T^2}{2})\ (C_3 + \delta C_4) > C_3\ (-\delta t_1 + \frac{\delta^2 t_1^2}{2})$

Multiplying by 2 on both sides of the above inequality

$\Rightarrow \delta T\ (\delta T - 2)\ (C_3 + \delta C_4) > \delta C_3\, t_1\ (\delta t_1 - 2)$

$\Rightarrow T\ (\delta T - 2)\ (C_3 + \delta C_4) > C_3\, t_1\ (\delta t_1 - 2)$

Hence the proof

Converse: let $\frac{\partial^2 C}{\partial T^2} > 0$ and we shall show that $(1-e^{-\delta T})$ will be positive.

We shall show $(1-e^{-\delta T}) > 0$ by contradiction

If $(1-e^{-\delta T}) < 0$

Multiplying by $e^{\delta T}$ on both sides of the above inequality, we get

$e^{\delta T} - 1 < 0$

$\Rightarrow e^{\delta T} < 1$

$\Rightarrow \left[1+\delta T + \frac{\delta^2 T^2}{2} + \cdots\right] < 1$

$\Rightarrow \delta\left[T + \frac{\delta T^2}{2} + \cdots\right] < 0$

Since multiple of $\delta > 0$ in the above inequality cannot be negative because it is the function of time which implies that $\delta < 0$, which will be impossible. Therefore our assumption is wrong. Hence the proof.

Theorem: 2. The cost function C (t_1, T) possess a minimum solution.

Proof: $C(t_1, T) = \frac{R}{T}$ where R is given in equation (14)

$$\frac{\partial C(t_1,T)}{\partial t_1} = \frac{1}{T}\Big[\frac{a\,D(p)}{2\theta}\{2(\theta\mu+1)\ e^{\theta(t_1-\mu)} - 2\} + \frac{D(p)\,b}{6\theta^3}\big[-6\theta^2 t_1 - 6\theta + 6\ \theta\, e^{\theta(t_1-\mu)}\{\mu\theta(\theta+1)+1\}\big] + D(p)\,C_2\{e^{\theta(t_1-\mu)} - 1\} + \frac{C_3 D(p)}{\delta^2}\{\delta\ e^{-\delta t_1} - T\ \delta^2 e^{-\delta t_1} - \delta\ e^{-\delta t_1} + \delta^2 t_1 e^{-\delta t_1}\} + \frac{C_4 D(p)}{\delta}\{\delta e^{-\delta t_1} - \delta\}\Big]$$

$$= \frac{D(p)}{T}\Big[\frac{a}{\theta}\{(\theta\mu+1)\,e^{\theta(t_1-\mu)} - 1\} + \frac{b}{\theta^2}\big[e^{\theta(t_1-\mu)}\{\mu\theta(\theta+1)+1\} - \theta t_1 - 1\big] + C_2\big[e^{\theta(t_1-\mu)} - 1\big] + e^{-\delta t_1}\{C_3(t_1 - T) + C_4\} - C_4\Big] \qquad \text{..... (17)}$$

$$\frac{\partial^2 C(t_1,T)}{\partial t_1^2} = \frac{D(p)}{T}\Big[a\{(\theta\mu+1)\,e^{\theta(t_1-\mu)}\} + \frac{b}{\theta}\big[e^{\theta(t_1-\mu)}\{\mu\theta(\theta+1)+1\} - 1\big] + C_2\big[\theta e^{\theta(t_1-\mu)}\big] + e^{-\delta t_1}\{C_3(t_1 - T) + C_4\} - C_4\Big]$$

Since $\delta \geq 0$, $t_1 \geq \mu \geq 0$, $T > t_1 \geq 0$ therefore we get

$$\frac{\partial^2 C(t_1,T)}{\partial t_1^2} > 0 \qquad \text{.... (18)}$$

Now

$$\frac{\partial C(t_1,T)}{\partial T} = -\frac{1}{T^2}R + \frac{1}{T}\frac{\partial R}{\partial T}$$

$$= -\frac{1}{T^2}R + \frac{1}{T}\Big[\frac{C_3 D(p)}{\delta}(e^{-\delta t_1} - e^{-\delta T}) + C_4 D(p)(1 - e^{-\delta T})\Big]$$

$$\text{..... (19)}$$

$$\frac{\partial^2 C(t_1,T)}{\partial T^2} = \frac{2R}{T^3} - \frac{2D(p)}{T^2}\Big[-\frac{C_3}{\delta}(e^{-\delta T} - e^{-\delta t_1}) + C_4(1 - e^{-\delta T})\Big] +$$

$$\frac{D(p)}{T}\left[C_3 e^{-\delta T} + C_4 \delta\, e^{-\delta T}\right] \qquad \text{….. (20)}$$

Since R will be positive and $T > t_1 \geq 0$, by lemma 4, we get

$$\frac{\partial^2 C\,(t_1, T)}{\partial\, T^2} > 0 \qquad (\because \text{ theorem 1}) \qquad \text{….. (21)}$$

$$\frac{\partial^2 C\,(t_1, T)}{\partial\, t_1 \partial\, T} = -\frac{D\,(p)}{T^2}\Big[\frac{a}{\theta}\left\{(\theta\,\mu + 1)\, e^{\theta\,(t_1-\mu)} - 1\right\} + \frac{b}{\theta^2}\left[e^{\theta\,(t_1-\mu)}\left\{\mu\,\theta\,(\theta+1)+1\right\} - \theta\, t_1 - 1\right]$$

$$+ C_2\left[e^{\theta\,(t_1-\mu)} - 1\right] + e^{-\delta\, t_1}\left\{C_3\,(t_1 - T) + C_4\right\} - C_4\Big] - \frac{D\,(p)}{T} C_3 e^{-\delta\, t_1} \text{… (22)}$$

The function $C\,(t_1, T)$ has the minimum solution with respect to t_1& T since Equation

$$\frac{\partial^2 C\,(t_1, T)}{\partial\, t_1^2} > 0 \;,\; \frac{\partial^2 C\,(t_1, T)}{\partial\, T^2} > 0$$

$$\left(\frac{\partial^2 C\,(t_1, T)}{\partial\, t_1^2}\right)\left(\frac{\partial^2 C\,(t_1, T)}{\partial\, T^2}\right) - \left(\frac{\partial^2 C\,(t_1, T)^2}{\partial\, t_1 \partial\, T}\right) > 0 \qquad \text{….. (23)}$$

is satisfied. Hence the proof

4. Analysis the lemmas for at least one value of decision variable t_1 & T:

By lemma 6.3 $\delta > \frac{2}{T + t_1}$,if we take $T + t_1 = 2$ since $T > t_1$

$\Rightarrow T = 2 - t_1$ so $2 - t_1 > t_1$

$\Rightarrow 1 > t_1$ but $t_1 \geq 0$

$\Rightarrow 0 \leq t_1 < 1$ that is, $t_1 \in [0, 1)$

We have assumed that $T > t_1$ but $T \neq t_1$ so $T > 1$ and T always be positive.

We can take demonstrative values as $T = \frac{3}{2}$, $t_1 = \frac{1}{2}$& $\delta = \frac{3}{2}$these values satisfy the lemmas and theorem.

5. Determination of optimal values $T^*$and t_1^*:

The roots of the equations (17) & (19) will give the optimal values $T = T^*$ & $t_1 = t_1^*$ Which minimize the

function C (t_1, T), provided they satisfy the sufficient conditions

$$\frac{\partial^2 C\,(t_1, T)}{\partial\, t_1^2} > 0 \quad , \quad \frac{\partial^2 C\,(t_1, T)}{\partial\, T^2} > 0$$

$$\left(\frac{\partial^2 C\,(t_1, T)}{\partial\, t_1^2}\right)\left(\frac{\partial^2 C\,(t_1, T)}{\partial\, T^2}\right) - \left(\frac{\partial^2 C\,(t_1, T)^2}{\partial\, t_1 \partial\, T}\right) > 0$$

If the solutions obtained from equations (17) & (19) do not satisfy the sufficient condition and may conclude that no feasible solution will be optimal for the set of parameter values taken to solve equations (17) and (18). Such a situation will imply that the parameter values are inconsistent and there is some error in their estimation.

6. Conclusion:

We have developed and analyze inventory models with variable holding cost as (a + b t) i.e. holding cost will be variable with respect to time. Where 'a' & 'b' are positive constants such that $0 < a < b \leq 1$ and $b \neq 0$. We observe that, when holding cost function will be increase linearly with very small values of 'a' and 'b' then average total cost function will also increase. Deterioration is taken into account only after the life time of item. Shortage in inventory is allowed with partially backlogged and backlogging rate is exponentially decreasing function of the waiting time for the next replenishment. δ is taken as backlogging parameter. We observed that, optimal values of t_1 & T will also satisfied by condition $T\,(\delta\, T - 2) > t_1(\delta\, t_1 - 2)$ for $\delta > 0$.We have developed a procedure for finding the optimal values of t_1 & T in certain interval by cost minimization technique.

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