RESEARCH ARTICLE OPEN ACCESS

Probability Inequality Relation between the Area of Given Triangle and that of its Inscribed Triangle

BISHAL PANTHI*AASHISH SHAH**

*Kathmandu Model Secondary School, Kathmandu, Nepal

Email: panthibishal211@gmail.com

**Dhanusa Science Campus, Dhanusha, Nepal

Email:shahaaseesh1@gmail.com

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Abstract:

In this research paper, we will derive the functional equation demonstrating the area of an inscribed triangle. Using relation of area between the various triangles formed in given triangle and associating it with 3-D co-ordinates system, we will derive the functional equation "xy+yz+zx=0". After that, it will be plotted in the graph and a probability of an inscribed triangle area being greater than or less than ¼ of given triangle will be derived. Here, we will define the standard inscribed triangle being formed by joining the mid-points of each sides of a triangle whose area is ¼ of the area of given triangle.

Keywords **—Inscribed Triangle, Double Integral, Probability, Inequality**

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I. INTRODUCTION

Suppose, we choose three points randomly on each sides of a triangle and join all those three points to form an inscribed triangle. Now, we are going to explore the relation of area between the inscribed triangle and given triangle. Here, the triangle formed by joining the mid-points of each sides of triangle is defined as standard inscribed triangle. The area of the standard triangle is $\frac{1}{4}$ of the given triangle. In respect to standard inscribed triangle, we will develop a probability inequality problem.

II.METHODOLOGY

Establishing functional relationship between the area of inscribed triangle and that of given triangle

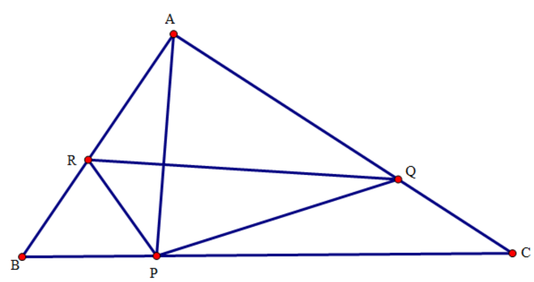

Figure 1:

In the given figure 1, P, Q and R are the random points on side BC, CA and AB respectively. Here, A and P are joined.

Let,

Area of $\Delta\mathbf{CPQ} = \boldsymbol{A_1}$

Area of $\Delta\mathbf{AQR} = \boldsymbol{A_2}$

Area of $\Delta\mathbf{BRP} = \boldsymbol{A_3}$

Let, $\lambda_1 = \frac{CQ}{CA} = \frac{A_1}{[\Delta\mathbf{APC}]}$.................... (1)

$\lambda_2 = \frac{AR}{AB}$

$\lambda_3 = \frac{BP}{BC}$

$\therefore 1-\lambda_3 = 1 - \frac{BP}{BC} = \frac{CP}{BC} = \frac{[\Delta\mathbf{APC}]}{[\Delta\mathbf{ABC}]}$ .... (2)

From (1) and (2), we get:

$$A_1 = \lambda_1[\triangle \mathbf{APC}]$$
$$= \lambda_1(1-\lambda_3)\,[\triangle \mathbf{ABC}]$$

Likewise,

$$A_2 = \lambda_2(1-\lambda_1)\,[\triangle \mathbf{ABC}]$$
$$A_3 = \lambda_3(1-\lambda_2)[\triangle \mathbf{ABC}]$$

Now:

$$[\triangle \boldsymbol{PQR}] = [\triangle \mathbf{ABC}] - (\boldsymbol{A_1} + \boldsymbol{A_2} + \boldsymbol{A_3})$$
$$= [\triangle \mathbf{ABC}] - [\triangle \mathbf{ABC}]\{\lambda_1(1-\lambda_3) + \lambda_2(1-\lambda_1) + \lambda_3(1-\lambda_2)\}$$

$$\frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} = 1 - \{\lambda_1(1-\lambda_3) + \lambda_2(1-\lambda_1) + \lambda_3(1-\lambda_2)\}$$

$$\therefore \frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} = \lambda_1\lambda_2\lambda_3 + (1-\lambda_1)(1-\lambda_2)(1-\lambda_3) \text{................ (I)}$$

If $\triangle \boldsymbol{PQR}$ is standard inscribed triangle, then

$$\frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} = \frac{1}{4}$$

Let us suppose,

$x = \lambda_1 - \frac{1}{2}$ .... (II)

$y = \lambda_2 - \frac{1}{2}$ .... (III)

$z = \lambda_3 - \frac{1}{2}$ .... (IV)

Then, from (I), (II), (III) and (IV), we get:

$$\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)\left(z+\frac{1}{2}\right) - \left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)\left(z-\frac{1}{2}\right) = 0$$

Generally, for any inscribed triangle, there are three cases:

1) xy+yz+zx<0
2) xy+yz+zx=0
3) xy+yz+zx>0

where, $-\frac{1}{2} \le x, y, z \le \frac{1}{2}$

Generally for any inscribed triangle of, there are three cases:

$$\frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} < \frac{1}{4} \text{ or } \frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} = \frac{1}{4} \text{ or } \frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]} > \frac{1}{4}$$

We will further establish the functional relationship between the positions of P,Q,R and ordered pair (x,y,z) in 3-D co-ordinates system.

III.RESULT

Using 3-D Model and double integral for analyzing probability inequality

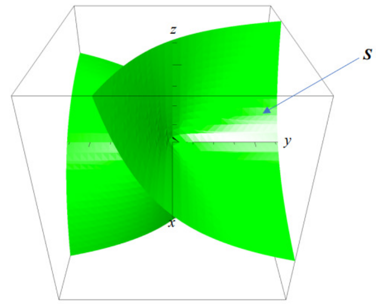

FIGURE 2:

The figure 1 is the graph of xy+yz+zx=0 ($-\frac{1}{2} \le x, y, z \le \frac{1}{2}$) in 3-D co-ordinates system. We will find the volume of S in cube with side length 1 unit by using double integral. So, we will get the probability that shows the area of random inscribed triangle $\triangle \boldsymbol{PQR}$ is greater than, equal to or less than $\frac{1}{4}$ of area of $\triangle \boldsymbol{ABC}$.

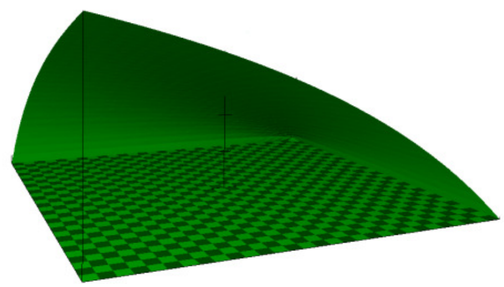

FIGURE 3:

The volume of S is formed by 3 spaces along with a cube of side length 0.5 units. From the equation of curve xy+yz+zx=0, we get:

$$x = \frac{-yz}{y+z}$$

After that, we will use the double integral to calculate the volume of S.

$$\text{Now, Volume of S} = -6\left\{\int_0^{\frac{1}{2}} \left(\int_0^z \frac{-yz}{y+z}\,dy\right) dz\right\} + \left(\frac{1}{2}\right)^3 \text{........................ (3)}$$

Let us separately calculate $\int_0^z \frac{-yz}{y+z}\,dy$

$$\int_0^z \frac{-yz}{y+z}\,dy = -z\int_0^z \frac{y+z}{y+z}\,dy + z^2\int_0^z \frac{1}{y+z}\,dy$$
$$= -z[y]_0^z + z^2[\ln(y+z)]_0^z$$
$$= -z^2 + z^2\ln 2$$

Again,

$$-\int_0^{\frac{1}{2}}(-z^2+z^2\,ln2)\,dz$$

$$=-\left\{-\left[\frac{z^3}{3}\right]_0^{\frac{1}{2}}+\ln 2\left[\frac{z^3}{3}\right]_0^{\frac{1}{2}}\right\}$$

$$=-\left(-\frac{1}{24}+\frac{ln2}{24}\right)$$

$$=\frac{1}{24}-\frac{ln2}{24}$$

∴The volume of S= $6\left(\frac{1}{24}-\frac{ln2}{24}\right)+\frac{1}{8}$

$$=\frac{1}{4}-\frac{ln2}{4}+\frac{1}{8}$$

$$=\frac{3}{8}-\frac{ln2}{4}$$

Now, the probability that $\frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]}>\frac{1}{4}=\frac{2\left(\frac{-ln2}{4}+\frac{3}{8}\right)}{1^3}$

$$\approx 0.4034264$$

On ignoring the thickness of curve xy+yz+zx=0, we can say that the probability of $\frac{[\triangle \boldsymbol{PQR}]}{[\triangle \mathbf{ABC}]}<\frac{1}{4}$ = 1- $\frac{2\left(\frac{-ln2}{4}+\frac{3}{8}\right)}{1^3}\approx 0.59657$

IV. CONCLUSIONS

We can conclude that the probability of inscribed triangle being area greater than ¼ of that of given triangle is nearly 0.4034264. Similarly, the probability of inscribed triangle area being less than ¼ of that of given triangle is nearly 0.596573. .

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