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RESEARCH ARTICLE OPEN ACCESS

INVESTIGATION OF SOLUTIONS TO AN EXPONENTIAL DIOPHANTINE EQUATION $p_1{}^x + p_2{}^y + p_3{}^z = M^2$ FOR PRIME TRIPLETS (p_1, p_2, p_3)

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Abstract:

In this article, the solutions to the Diophantine equation $p_1{}^x + p_2{}^y + p_3{}^z = M^2$ where (p_1, p_2, p_3) is a prime triplet of the forms $(p, p+2, p+6)$ and $(p, p+4, p+6)$ for x, y, z are integers takes the values of1 or 2 is investigated by applying the basic concepts of Mathematics. Also, few choices of x, y, z are not possible solutions of the equation is confirmed by MATLAB Program.

Keywords **-- Diophantine equation, integer solutions**

I. INTRODUCTION

Primeval and wide-ranging, the discipline of Diophantine equations lacks a standardised approach for determining if an equation has any solutions or how many solutions. Numerous variations of the well-known general equation $p^x + q^y = z^2$emerge. It is possible to locate a substantial quantity of literature on non-linear equations using specific primes and powers of numerous kinds.With varied degrees of success, numerous writers [1,2,5,6,7] have revisited the above problem and tried many primes, such as the Mersenne prime, in an effort to solve it.In [3,4], Nechemia Burshteinprotracted the above equation as $p^x + (p+1)^y + (p+2)^z = M^2$ and $p^x + (p+1)^y + (p+2)^z = M^3$, for all primes p and for specific powers ofx, y, z and the findings were supported by simple mathematical tools

If there are three prime numbers in the set and the smallest and biggest vary by six places, then the set is called a prime triplet. To be precise, the sets must be $(p, p + 2, p + 6)$ or $(p, p + 4, p + 6)$ in form. This is the closest conceivable collection of three prime numbers, with the exception of (2, 3, 5) and (3, 5, 7).

In this article, the Diophantine equation $p_1{}^x + p_2{}^y + p_3{}^z = M^2$ is established and results are analyzed for the prime triplets where p is of the form $4n + 1$or $4n + 3$and the powers of primes are either 1 or 2.p

II. METHODOF EXTRACTING INTEGER SOLUTIONS

The approach of existence ofinteger solutions to an equation$p_1{}^x + p_2{}^y + p_3{}^Z = M^2$where (p_1, p_2, p_3) is a prime 3-tuples is analyzed in the following theorems.

Theorem 2.1

If $x, y, z \in \{1,2\}$ and $(p, p + 2, p + 6)$ is a prime triplet of the form $(4n + 1, 4n + 3, 4n + 7)$, $n \in \mathcal{N}$ then an equation $p^x + (p + 2)^y + (p + 6)^Z = M^2$ has no solution.

Proof:

The theorem is proved by considering the following eight cases.

Case 1: $x = 1, y = 1, z = 1$

Then,$p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow 4n + 1 + 4n + 3 + 4n + 7 = M^2$$

$$\Rightarrow 12n + 11 = M^2$$

It is scrutinized that the expression $12n + 11$ is not a perfect square for any $n \in \mathcal{N}$.

The following MATLAB Programdemonstrates the statement given above.

```
clc; clear all;
n = input('Enter a natural number n');
for i = 1:n
    p1 = 4 * i + 1; p2 = 4 * i + 3; p3 = 4 * i + 7;
    if(isprime(p1) == 1 & isprime(p2) == 1 & isprime(p3) == 1)
        MS = 12 * n + 11;
        M = sqrt(MS);
        if(rem(M, 1) == 0)
            fprintf('p1 = %d, p2 = %d, p3 = %d, M = %d', p1, p2, p3, M)
        end
```


end

end

Case2:$x = 2, y = 1, z = 1$

The equation to analyze solutions in integers can be written as

$$(4n + 1)^2 + (4n + 3) + (4n + 7) = M^2$$

$$\Rightarrow 16n^2 + 16n + 11 = M^2$$

$$\Rightarrow (4n + 2)^2 + 7 = M^2$$

$$\Rightarrow M^2 - (4n + 2)^2 = 7$$

This is possible only when $M = 4$ and $4n + 2 = 3$. But no such $n \in \mathcal{N}$ satisfying the equation $4n + 2 = 3$.

Hence, there exists no integer solution.

Case3:$x = 1, y = 2, z = 1$

For these choices of x, y, z, the original equation is reduced into

$$(4n + 1) + (4n + 3)^2 + (4n + 7) = M^2$$

$$\Rightarrow 16n^2 + 32n + 17 = M^2$$

$$\Rightarrow (4n + 4)^2 + 1 = M^2$$

$$\Rightarrow M^2 - (4n + 4)^2 = 1$$

It is well-known that, difference of two square number cannot be 1.

Therefore, this case does not yield a solution.

Case 4 : $x = 1, y = 1, z = 2$

The selected values of the variables convert the given equation as follows

$$(4n + 1) + (4n + 3) + (4n + 7)^2 = M^2$$

$$\Rightarrow 16n^2 + 64n + 53 = M^2$$

$$\Rightarrow (4n + 8)^2 - 11 = M^2$$

$$\Rightarrow (4n + 8)^2 - M^2 = 11$$

This is true only if $M = 5$ and $4n + 8 = 6$.But for any $n \in N$, $4n + 8 = 6$ is not valid.

Consequently, there is no solution to an equation.

Case 5 :$x = 2, y = 2, z = 1$

The desired equation becomes

$$(4n + 1)^2 + (4n + 3)^2 + (4n + 7) = M^2$$

$$\Rightarrow 32n^2 + 36n + 17 = M^2$$

Case 6 :$x = 2, y = 1, z = 2$

The considered equation becomes

$$(4n + 1)^2 + (4n + 3) + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 68n + 53 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

Then,$p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow (4n + 1) + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 84n + 59 = M^2$$

Case 8:$x = 2, y = 2, z = 2$

Then $p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow (4n + 1)^2 + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 48n^2 + 88n + 59 = M^2$$

It is a well-known fact that if $b^2 = 4ac$, the quadratic polynomial $ax^2 + bx + c$ is a perfect square.

But, the quadratic equation in $n$mentioned above from case 5 to case 8does not meet this criterion. As a conclusion, none of these choices of x, y, z considered from case 5 to case 8 provides solutions to an equation.

Theorem 2.2

A solution to the equation$p^x + (p + 2)^y + (p + 6)^Z = M^2$ is inconceivable if $x, y, z \in \{1,2\}$ and $(p, p + 2, p + 6)$ is a prime triplet of the form $(4n + 3, 4n + 5, 4n + 9)$.

Proof:

This theorem is showed by the succeeding eight cases as in theorem 2.1

Case 1: $x = 1, y = 1, z = 1$

Then,$p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow 4n + 3 + 4n + 5 + 4n + 9 = M^2$$

$$\Rightarrow 12n + 17 = M^2$$

This is not true for any $n \in \mathcal{N}$. This statement is confirmed by the succeeding MATLAB Program.

$$clc;\ clear\ all;$$

$$n = input('Enter\ a\ natural\ number\ n');$$

$for\ i = 1{:}\,n$

$p1 = 4 * i + 3;\ p2 = 4 * i + 5;\ p3 = 4 * i + 9;$

$if(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 11;$

$M = sqrt(MS);$

$if(rem(M, 1) == 0)$

$fprintf('p1 = \%d, p2 = \%d, p3 = \%d, M = \%d', p1, p2, p3, M)$

end

end

end

Case 2:$x = 2, y = 1, z = 1$

The required equation to be solved becomes

$$(4n + 3)^2 + (4n + 5) + (4n + 9) = M^2$$
$$\Rightarrow 16n^2 + 32n + 23 = M^2$$
$$\Rightarrow (4n + 4)^2 + 7 = M^2$$
$$\Rightarrow M^2 - (4n + 4)^2 = 7$$

This declaration is true only when $M = 4$ and $4n + 4 = 3$. But there is no $n \in \mathcal{N}$ sustaining the condition$4n + 4 = 3$.

Case 3:$x = 1, y = 2, z = 1$

The developed equation can be modified into

$$(4n + 3) + (4n + 5)^2 + (4n + 9) = M^2$$
$$\Rightarrow 16n^2 + 48n + 37 = M^2$$
$$\Rightarrow (4n + 6)^2 + 1 = M^2$$
$$\Rightarrow M^2 - (4n + 6)^2 = 1$$

As is case 2 of theorem 2.1, this is impossible.

Case 4 : $x = 1, y = 1, z = 2$

The given equation can be rewritten as

$$(4n + 3) + (4n + 5) + (4n + 9)^2 = M^2$$
$$\Rightarrow 16n^2 + 80n + 89 = M^2$$
$$\Rightarrow (4n + 10)^2 - 11 = M^2$$
$$\Rightarrow (4n + 10)^2 - M^2 = 11$$

$\Rightarrow M = 5$ and $4n + 10 = 6$

But for any $n \in N$, $4n + 10 = 6$ is not possible.

Case 5 :$x = 2, y = 2, z = 1$

The stated equation becomes

$$(4n+3)^2 + (4n+5)^2 + (4n+9) = M^2$$

$$\Rightarrow 32n^2 + 68n + 43 = M^2$$

Case 6 :$x = 2, y = 1, z = 2$

The considered equation is

$$(4n+3)^2 + (4n+5) + (4n+9)^2 = M^2$$

$$\Rightarrow 32n^2 + 100n + 95 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

These options of the variables reduce the scrutinized equation into

$$(4n+3) + (4n+5)^2 + (4n+9)^2 = M^2$$

$$\Rightarrow 32n^2 + 116n + 109 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

The equation in which solutions to be discovered becomes

$$(4n+3)^2 + (4n+5)^2 + (4n+9)^2 = M^2$$

$$\Rightarrow 48n^2 + 136n + 115 = M^2$$

As in theorem 2.1, in this theorem also case 5 to case 8 does not yield the solution to an equation. Hence, there exists no solution in integer to the given equation.

Theorem 2.3

There are infinitely many solutions to the equation $p^x + (p+4)^y + (p+6)^Z = M^2$ if $(p, p+4, p+6)$ is a prime triplet the form $(4n+1, 4n+5, 4n+7), n \in \mathcal{N}$ and if x, y, z are either of 1 or 2.

Proof:

The theorem is proved as in previous two theorems.

Case 1: $x = 1, y = 1, z = 1$

Then,$p^x + (p+4)^y + (p+6)^Z = M^2$

$$\Rightarrow 4n + 1 + 4n + 5 + 4n + 7 == M^2$$

$$\Rightarrow 12n + 13 = M^2$$

It is observedfrom the following MATLAB Program,there are enormous prime triplets can be extracted as a solution. For instance, if$n = 3, 9, 69, 153$provides the prime triplets $(13, 17, 19), (31, 41, 43), (277, 281, 283), (613, 617, 619)$ as solutions to the designated equation.

```
clc; clear all;
n = input('Enter a natural number n');
for i = 1:n
    p1 = 4*i + 1; p2 = 4*i + 5; p3 = 4*i + 7;
    if(isprime(p1) == 1 & isprime(p2) == 1 & isprime(p3) == 1)
        MS = 12*n + 13;
        M = sqrt(MS);
        if(rem(M, 1) == 0)
            fprintf('p1 = %d, p2 = %d, p3 = %d, M = %d', p1, p2, p3, M)
        end
    end
end
```

Case 2:$x = 2, y = 1, z = 1$

The given equation becomes

$$p^x + (p + 4)^y + (p + 6)^Z = M^2$$

$$\Rightarrow (4n + 1)^2 + (4n + 5) + (4n + 7) = M^2$$

$$\Rightarrow 16n^2 + 16n + 13 = M^2$$

$$\Rightarrow (4n + 2)^2 + 9 = M^2$$

$$\Rightarrow M^2 - (4n + 2)^2 = 9$$

This is achievable only when $M = 5$ and $4n + 2 = 4$. However, for every $n \in \mathcal{N}$, the equation$4n + 2 = 4$ is not feasible.

Case 3:$x = 1, y = 2, z = 1$

The elected choices of x, y, z minimizes the given equation as

$$(4n + 1) + (4n + 5)^2 + (4n + 7) = M^2$$

$$\Rightarrow 6n^2 + 48n + 33 = M^2$$

$$\Rightarrow (4n + 6)^2 - 3 = M^2$$

$$\Rightarrow (4n + 6)^2 - M^2 = 3$$

$\Rightarrow 4n + 6 = 2$ and $M = 1$are the only values that enable the above equation to be accomplished. But $4n + 6 = 2$ is not conceivable for any $n \in \mathcal{N}$.

Case 4 : $x = 1, y = 1, z = 2$

For these options of x, y, z, the equation to be resolved is

$$(4n + 1) + (4n + 5) + (4n + 7)^2 = M^2$$

$$\Rightarrow 16n^2 + 64n + 55 = M^2$$

$$\Rightarrow (4n + 8)^2 - 9 = M^2$$

$$\Rightarrow (4n + 8)^2 - M^2 = 9$$

The only values which attain the above condition are $4n + 8 = 5$ and $M = 4$.

But for any $n \in \mathcal{N}$, $4n + 8 = 5$ is invalid.

Case 5 :$x = 2, y = 2, z = 1$

Therefore, the original equation is converted into the quadratic equation as follows

$$32n^2 + 52n + 33 = M^2$$

Case 6 :$x = 2, y = 1, z = 2$

Then, the original equation is altered into the quadratic equation in n as given below.

$$32n^2 + 68n + 55 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

The similar form of the given equation is

$$32n^2 + 100n + 75 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

The identical from of the considered equation is

$$48n^2 + 104n + 75 = M^2$$

As the explanation given in theorem 2.1, there is no solution in integers for the cases listed above from 5 to 8.

Theorem 2.4:

For any $n \in \mathcal{N}$, if $p = 4n + 3$ and $(p, p + 4, p + 6)$ is a prime triplet, then$p^x + (p + 4)^y + (p + 6)^Z = M^2$ has no solution when x, y, z are either 1 or 2.

Proof:

The proof is analogous to theorem 2.1

Exceptional Prime Triplets

1. If $(p_1, p_2, p_3) = (2, 3, 5)$, then the possible solution of $2^x + 3^y + 5^z = M^2$ are $(x, y, z, M) = (1, 2, 1, 4)$ and $(1, 2, 2, 6)$.
2. If $(p_1, p_2, p_3) = (3, 5, 7)$, then there is no solution to the proposed equation $3^x + 5^y + 7^z = M^2$

III. CONCLUSION

This text investigates the spectacular exponential Diophantine equation ${p_1}^x + {p_2}^y + {p_3}^z = M^2$ where (p_1, p_2, p_3) is a prime triplet either of the form $(p, p + 2, p + 6)$ or $(p, p + 4, p + 6)$ and x, y, z are either 1 or 2.One may derive integer solutions by considering the sum of the variables or the product of the variables is either 1 or 2.

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