RESEARCH ARTICLE OPEN ACCESS

TAKE OFF AND LANDING PERFORMANCE OF AN AIRCRAFT WITH CONSTANT/VARIABLE THRUST

SN MAITRA,

Retired Head of Mathematics Department, National Defense Academy, Khadakwasla, Pune-411023, India.

ABSTRACT:

The paper pertains to take off and landing performance of an aircraft or a rocket-powered vehicle. The velocity acquired and distance traveled by the aircraft on the runway at any instant of time during take- off and during landing are determined considering variation of its weight due to its fuel consumption. Motion of the aircraft during take- off with constant thrust and during landing with reversed thrust are dealt with and finally during landing by applying thrust equal to drag.

INTRODUCTION

Angelo Miele[1] (1962) framed equations of motion of an aircraft for take-off and landing performance on a horizontal runway and solved them in closed form but without taking into consideration variation of its weight due to fuel consumption. In this design, however small may be the fuel consumption during either of the runs variation of the weight due to fuel consumption is taken into account in order to determine the velocity attained and distance traveled on the runway.DJ Kettle[2] described ground performance at take- off and landing. The equations of motion of the aircraft are represented[1] as

$$\dot{x} = v$$

$$T - D - \mu R - \frac{W}{g}\dot{v} = 0 \qquad (1)$$

$$R - W + L = 0$$

Since in this feature fuel consumption during its travel on the ground is not neglected in tackling motion of the aircraft another equation arises:

$$\dot{W} + cT = 0 \qquad (2)$$

where x denotes the distance described on the horizontal runway ,v the velocity, T the thrust,D the drag, W the weight after a run for time t during landing or take- off ,L the lift,R the normal reaction exerted by the
runway, μ the coefficient of friction , W_0 initial weight for take off/

landing , c the specific fuel consumption and the dot sign a derivative with respect to time t. The atmospheric drag[1] and lift[1] are respectively :

$$D = \frac{1}{2} C_D \rho s v^2 \quad \text{and} \quad L = \frac{1}{2} C_L \rho s v^2 \qquad (3)$$

where C_D and C_L are respectively the drag and lift **coefficients, ρ the air density, S the reference area of the aircraft.**

MOTION ON THE RUNWAY WITH CONSTANT THRUST

Initial and boundary conditions are prescribed:

At time t=0, $W = W_0$, v=0 ,x=0 for take- off, where the distance traveled in time t_k to take off is x_k, whereas

At time t=0, $W = W_0$, $v = v_L$ = touch down velocity and x=0 in the beginning of the landing and on completion of landing, ie, at $t = t_L$, $v = 0$, $x = x_L$ (4)
In other words the distance traveled on the runway in coming to rest in time t_L of landing is x_L. Because of constant thrust and constant specific fuel consumption, the solution to equation (2) by virtue of the initial conditions (4) gives

$$W = W_0 - cTt \qquad (5)$$

Since the fuel consumption during movement of the aircraft along the runway for a short time is small compared to the fuel intake ,we rule out choosing the instantaneous weight of the aircraft as the independent variable for solutions to the governing differential equations of motion . Combining (1), (2),(5) and (3) is obtained

$$\frac{dt}{dv} = \frac{W_0 - cTt}{g\{T + \mu cTt - \frac{1}{2}\rho S(C_D - \mu C_L)v^2 - \mu W_0)\}}$$

Now the term involving t in the denominator of this equation being small,is neglected in that the fuel consumption for this short time is small while the similar term in the numerator is retained without loss of sufficient accuracy and generality. Hence

$$\frac{dt}{dv} = \frac{W_0 - cTt}{g\{T - \frac{1}{2}\rho S(C_D - \mu C_L)v^2 - \mu W_0)\}} \qquad \mu \leq 1$$

which can be reproduced as

$$\frac{dt}{dv} = \frac{k(b-t)}{a^2 - v^2} \qquad (a>v) \qquad (6)$$

where $k = \frac{2cT}{g\rho S(C_D - \mu C_L)}$, $a^2 = 2\left(\frac{T - \mu W_0}{\rho S(C_D - \mu C_L)}\right) > 0$ and $b = \frac{W_0}{cT}$ (6.1)

Integral of (6) subject to the initial conditions (4) yields take- off velocity v_k in terms of the time taken t_k:

$$\int_0^{t_k} \frac{dt}{b-t} = k\int_0^{v_k} \frac{dv}{a^2-v^2}$$

Or, $-\log(1-\frac{t_k}{b}) = \frac{k}{2a}\log\frac{a+v_k}{a-v_k} = -\frac{k}{2a}\log\frac{a-v_k}{a+v_k}$

$$t_k = b[1-(\frac{a-v_k}{a+v_k})^p]$$

where any time t during takeoff is

$$t = b[1-(\frac{a-v}{a+v})^p] \qquad (6.2)$$

Or equivalently $v_k = a\frac{1-(1-\frac{t_k}{b})^q}{1+(1-\frac{t_k}{b})^q}$ $\quad p=\frac{k}{2a}$, $q=1/p$ (7)

The velocity at any time t during its motion on the runway is obtained from (7) replacing v_k by v and t_k by t and as such to find the distance x traveled in time t on the runway , one can write

$$\frac{dx}{dt} = v = a\frac{1-(1-\frac{t_k}{b})^q}{1+(1-\frac{t_k}{b})^q} = a\{\frac{2}{1+(1-\frac{t_k}{b})^q} - 1\} \qquad (8)$$

Integrating which subject to the same initial conitions (4) the distance traveled before leaving the runway to take- off is given by

$$x_k = a\{2\int_0^{t_k} \frac{dt}{1+(1-\frac{t}{b})^q} - t_k\}$$

Putting $1-\frac{t}{b} = \tau$ so that $\tau_k = 1-\frac{t_k}{b}$ and $\tau_0 = 1$ at t=0 (9)

$$x_k = a[2b\int_{\tau_k}^1 \frac{d\tau}{1+\tau^q} - t_k] \;; \quad t \le b \text{ implies } \tau \le 1 \qquad (10)$$

For $\tau^q \le 1$, expanding binomially one $gets$

$$x_k = a[2b\int_{\tau_k}^1 \sum_{r=0}^{\infty}(-1)^r (\tau^q)^r d\tau - t_k]$$

$$x_k = a[b\sum_{r=0}^{\infty}(-1)^r \frac{\tau^{qr+1}}{qr+1}|_{\tau_k}^1 - t_k] \qquad (11)$$

Given some numerical integral values of q or p,the integral (10) can be evaluated in closed form. The distance traveled can be found as a function of velocity using (6) and initial conditions (4):

$$\frac{dx}{dv} = \frac{kv(b-t)}{a^2-v^2} = kb(\frac{a-v}{a+v})^p\frac{v}{a^2-v^2} \qquad [\text{ because of (6)}] \qquad (12)$$

Or, $x_k = kb\int_0^{v_k} (\frac{a-v}{a+v})^p \frac{vdv}{a^2-v^2}$ (By integrating by parts) (13)

$= \frac{-kb}{2ap}\{v_k(\frac{a-v_k}{a+v_k})^p - \int_0^{v_k}(\frac{a-v}{a+v})^p dv\}$

Because, $\int_o^{v_k}(\frac{a-v}{a+v})^p \frac{dv}{a^2-v^2} = \frac{-1}{2ap}\int_0^{v_k}\frac{d(\frac{a-v}{a+v})^p}{dv}dv = \frac{-1}{2ap}(\frac{a-v}{a+v})^p$

Hence, while tackling integral (13) by parts choosing v as the first form and

$(\frac{a-v}{a+v})^p \frac{dv}{a^2-v^2}$ as the second form is justified to arrive at the result((13)

$x_k = \frac{kb}{2ap}[I_k - v_k(\frac{a-v_k}{a+v_k})^p]$ (14)

where $I_k = \int_0^{v_k}(\frac{a-v}{a+v})^p dv = \int_0^{v_k}(\frac{2a}{a+v} - 1)^p dv$ which can be recast as

$I_k = \int_a^{a+v_k}(\frac{2a}{u})^p\left(1-\frac{u}{2a}\right)^p du$ (15)

with substitution u=a+v<2a because of (11). So expanding binomially in equation (17), we get

$$I_k = \int_a^{a+v_k}(\frac{2a}{u})^p\sum_{r=0}^{\infty}\frac{(-1)^r}{<r}p(p-1)(p-2)\ldots\ldots(p-r+1)(\frac{u}{2a})^r du$$

$I_k = [\sum_{r=0}^{\infty}\frac{(-1)^r}{<r(r-p+1)(2a)^{r-p}}p(p-1)(p-2)\ldots\ldots(p-r+1)(u)^{r-p+1}]_a^{v_k+a}$ (16)

Applying the similar treatment as above and the initial conditions (4) we can find the duration of motion of the aircraft on the runway after landing with reversal thrust or holding the thrust to drag at all times. The governing equations of motion for landing performance with reversed constant thrust are obtained by replacing T by−T in the denominator (6):

$\frac{dt}{dv} = \frac{-(W_0-cTt)}{g\{T+\frac{1}{2}\rho S(C_D-\mu C_L)v^2+\mu W_0)\}} = \frac{-k(b-t)}{d^2+v^2}$ (17)

where $d^2 = \frac{2(T+\mu W_0)}{\rho S(C_D-\mu C_L)v^2}$ but b and k have the same values as for take-off.

In consequence of the initial conditions (4), solution to (17) turns out to be

$Log(1-\frac{t}{b}) = \frac{k}{d}(\tan^{-1}\frac{v}{d} - \tan^{-1}\frac{v_L}{d}) = -\frac{k}{d}(\tan^{-1}\frac{v_L}{d} - \tan^{-1}\frac{v}{d})$ $(v_L > v)$

Or, $t = b\{1 - e^{-\frac{k}{d}(\tan^{-1}\frac{v_L}{d} - \tan^{-1}\frac{v}{d})}\}$ (18)

Duration of landing t_L is obtained by putting v=0 in the above equation;

$\mathrm{Log}(1-\frac{t_L}{b})=\frac{-k}{d}\tan^{-1}\frac{v_L}{d}$

Or, $t_L = b(1-e^{\frac{-k}{d}\tan^{-1}\frac{v_L}{d}}))$ (19)

From equation (18) velocity v can be expressed in terms of t so that

$\mathrm{Log}(1-\frac{t}{b}) = \frac{k}{d}(\tan^{-1}\frac{v}{d} - \tan^{-1}\frac{v_L}{d}) = \frac{k}{d}\tan^{-1}\frac{\frac{v}{d}-\frac{v_L}{d}}{1+\frac{v}{d}.\frac{v_L}{d}}$

Or, $\frac{\frac{v}{d}-\frac{v_L}{d}}{1+\frac{v}{d}.\frac{v_L}{d}} = \tan\{\frac{d}{k}\mathrm{Log}\left(1-\frac{t}{b}\right)\}$

$t=b(1-e^{\frac{k}{d}(\tan^{-1}\frac{v}{d}-\tan^{-1}\frac{v_L}{d})}) = b\,(1-e^{-\frac{k}{d}\tan^{-1}\frac{v_L}{d}}.e^{\frac{k}{d}\tan^{-1}\frac{v}{d}})$ (20))

$\frac{v}{d}\{1-\frac{v_L}{d}\tan\left\{\frac{d}{k}\mathrm{Log}\left(1-\frac{t}{b}\right)\right\} = \frac{v_L}{d} + \tan\{\frac{d}{k}\mathrm{Log}\left(1-\frac{t}{b}\right)\}$

Or, $v=\frac{dx}{dt} = d\frac{v_L + d\tan\{\frac{d}{k}\mathrm{Log}\left(1-\frac{t}{b}\right)\}}{\{d - v_L\tan\left\{\frac{d}{k}\mathrm{Log}\left(1-\frac{t}{b}\right)\right\}} = d.\frac{v_L - d\tan\frac{dt}{kb}}{d+v_L\tan\frac{dt}{kb}}$ (20.1)

For $\frac{t}{b} \ll 1$, are neglected the square and other higher powers of $\frac{dt}{kb}$ in logarithmic expansion in equation(20); one gets (For example log(1−Y)= −y where y<<1).Now

$v=\frac{dx}{dt}= \; = (\frac{d^2}{v_L})\frac{\frac{v_L}{d}\cos\frac{td}{bk}-\sin\frac{td}{bk}}{\frac{d}{v_L}\cos\frac{td}{bk}+\sin\frac{td}{bk}}$ (21)

$\frac{dx}{dt}=bk\frac{(\frac{d}{bk}\cos\frac{td}{bk}-\frac{d^2}{bkv_L}\sin\frac{td}{bk}}{\frac{d}{v_L}\cos\frac{td}{bk}+\sin\frac{td}{bk}}=bk$

Integrating(21) subject to the initial conditions(4) we get landing distance covered in time t:

$x=bk\log(\cos\frac{td}{bk}+\frac{v_L}{d}\sin\frac{td}{bk})$ (22)

We can also express distance x in terms of velocity v and hence

Eliminating t between (17) and (18) ,one gets

$\frac{dx}{dv} = \frac{-kAv}{d^2+v^2}e^{\frac{k}{d}\tan^{-1}\frac{V}{d}}$ (23)

where $A=be^{-\frac{k}{d}\tan^{-1}\frac{v_L}{d}}$

Integrating(23)by parts and using the initial and boundary conditions (4),we find the distance x_L travelled before coming to rest on the runway:

$$x_L = \frac{kA}{2}\{e^{\frac{k}{d}\tan^{-1}\frac{V}{d}}\log(d^2+v^2)|_0^{v_L} - kI_L\} \quad (24)$$

where $I_L = \int_0^{v_L} e^{\frac{k}{d}\tan^{-1}\frac{V}{d}} \frac{\log(d^2+v^2)}{d^2+v^2} dv$ (25)

To evaluate integral (25) let us put $\tan^{-1}\frac{V}{d} = z$ so that v=dtanz,

$v_L = d\tan z_L$ and $dv = d\sec^2 z dz$ (26)

Then $I_L = \frac{2}{d}\int_0^{z_L}\{\log(d\sec z)\}e^{\frac{kz}{d}} dz$ (27)

$=\frac{2\log d}{k}\left(e^{\frac{k}{d}z_L} - 1\right) - 2I_L^1$ $\quad \log(d\sec z) = \log d + \log\sec z$

where $I_L^1 = \int_0^{z_L}\{\log(\cos z)\}e^{\lambda z} dz$; $\lambda = \frac{k}{d}$ (28)

Integral (28) can be tackled by term-by- term integration after expanding logcosz in infinite series. Therefore let us begin with

$\mathrm{Log}\{(1+e^{2i\theta)})(1+e^{-2i\theta)})\} = \log(2+e^{2i\theta}+e^{-2i\theta})$

Interchanging the sides and expanding in logarithmic series,

$\mathrm{Log}(2+2\cos 2\theta) = \mathrm{Log}(1+e^{2i\theta)}) + \mathrm{Log}(1+e^{-2i\theta)})$

$$\log(4\cos^2\Theta) = e^{2i\theta} + \frac{e^{4i\theta}}{2} + \frac{e^{6i\theta}}{3} + \cdots \text{add infinity} + e^{-2i\theta} + \frac{e^{-4i\theta}}{2} + \frac{e^{-6i\theta}}{3} + \cdots \text{add infinity}$$

$$2\log 2 + 2\log\cos\theta = e^{2i\theta} + e^{-2i\theta} + \frac{e^{4i\theta}+e^{-4i\theta}}{2} + \frac{e^{6i\theta}+e^{-6i\theta}}{3} - \ldots \quad \text{add infinity}$$

$$\text{Or,} \quad \log\cos\theta = -\log 2 + \cos 2\theta + \frac{\cos 4\theta}{2} + \frac{\cos 6\theta}{3} + \frac{\cos 8\theta}{4} \ldots \quad \text{add } \infty \quad (29)$$

Replacing θ by z in (29)and then combining with (28)one gets

$$I_L^1 = \int_0^{z_L}[-e^{\lambda z}\log 2 + \sum_{r=1}^{\infty}\frac{1}{r}(\cos 2rz)e^{\lambda z}]\, dz$$

$$=(1-e^{\lambda z_L})(\log 2)/\lambda + \sum_{r=1}^{\infty}\frac{e^{\lambda z_L}(\lambda\cos 2rz_L + +2r\sin rz_L) - \lambda}{r(\lambda^2+4r^2)} \quad (30)$$

LANDING WITH THRUST EQUAL TO DRAG

In the light of the foregoing analysis ,if the thrust is equated to the drag(T=D), with due approximation variation of the weight due to fuel consumption is neglected with a view to finding only velocity and distance without sacrifice more or less accuracy coupled with controlled maneuvering of the aircraft. Hence the relevant equation is

$$\frac{dt}{dv} = \frac{-W_0}{g\mu\{W_0 - \frac{1}{2}C_L\rho S v^2)} = \frac{-2W_0}{g\mu\{2W_0 - C_L\rho S v^2)}$$

$$\text{Or, } \frac{dt}{dv} = \frac{-2W_0}{g\mu C_L\rho S\{\frac{2W_0}{C_L\rho S} - v^2)} \qquad (2W_0 > C_L\rho S v^2) \quad (31)$$

Integrating L.H.S. with respect to t from 0 to t and R.H.S. with respect to v from v_L to v are obtained

$$t = \frac{1}{2g\mu}\sqrt{\frac{2W_0}{C_L\rho s}}\log\left\{\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}} - v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}} + v\right)}\frac{\sqrt{2\frac{W_0}{C_L\rho s}} + v_L}{\sqrt{2\frac{W_0}{C_L\rho s}} - v_L}\right\} \qquad (32)$$

which exhibits time t as an explicit function of velocity v .We can also express v in terms of time t:

$$\frac{(\sqrt{2\frac{W_0}{C_L\rho s}} - v)}{(\sqrt{2\frac{W_0}{C_L\rho s}} + v)} = \frac{\sqrt{2\frac{W_0}{C_L\rho s}} - v_L}{\sqrt{2\frac{W_0}{C_L\rho s}} + v_L}\, e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})} \qquad (33)$$

$$\text{Or, } \frac{\sqrt{2\frac{W_0}{C_L\rho s}}}{v} = \frac{1 + B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}}{1 - B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}} \qquad \text{Or, } v = \frac{dx}{dt} = \{(\sqrt{2\frac{W_0}{C_L\rho s}})\frac{1 - B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}}{1 + B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}}\} \qquad (34)$$

$$\text{where} \quad B = \frac{\sqrt{2\frac{W_0}{C_L\rho s}} - v_L}{\sqrt{2\frac{W_0}{C_L\rho s}} + v_L} < 1 \qquad (34.1)$$

To facilitate integration of (34),we rearrange the same in the form

$$\frac{dx}{dt} = v = \sqrt{2\frac{W_0}{C_L\rho s}})\{1 - \frac{2B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}}{1 + B\,e^{(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}})}}\}$$

which is solved applying the initial conditions(4):

$$x=\left[\left(\sqrt{2\frac{W_0}{C_L\rho s}}\right)\left\{t-\frac{1}{g\mu}\sqrt{2\frac{W_0}{C_L\rho s}}\quad\log\frac{1+B\,e^{\left(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}}\right)}}{1+B}\right\}\right] \tag{35}$$

$$=2\frac{W_0}{C_L\rho sg\mu}\left\{\frac{1}{2}\log\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)}\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}-\log\frac{1+B\,e^{\log\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}}}{1+B}\right\}$$

$$=2\frac{W_0}{C_L\rho sg\mu}\left\{\frac{1}{2}\log\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)}\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}-\log\frac{1+\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)}}{1+\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v_L\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v_L\right)}}\right\} \quad =\frac{W_0}{C_L\rho sg\mu}\left\{\log\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)}\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}-2\log\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}+v}\right\}$$

$$=\frac{W_0}{C_L\rho sg\mu}\log\left\{\frac{\left(\sqrt{2\frac{W_0}{C_L\rho s}}-v\right)}{\left(\sqrt{2\frac{W_0}{C_L\rho s}}+v\right)}\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}\left(\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v}{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}\right)^2\right\}$$

$$x=\frac{W_0}{C_L\rho sg\mu}\log\frac{\left(\frac{W_0}{C_L\rho s}-v^2\right)}{\frac{W_0}{C_L\rho s}-v_L{}^2} \tag{36}$$

The velocity−distance relationship (36) can also be established directly by using equation (31) in a simple manner:

$$\frac{dt}{dv}\frac{dx}{dt}=\frac{-2W_0v}{g\mu C_L\rho S\{\frac{2W_0}{C_L\rho S}-v^2)}$$

$$\frac{dx}{dv}=\frac{-2W_0v}{g\mu C_L\rho S\{\frac{2W_0}{C_L\rho S}-v^2)}$$

which is integrated within the limits :v from v_L to v and x from o to x to obtain the same result as (36);

$$x=\frac{W_0}{C_L\rho sg\mu}\left\{\log\frac{\left(\frac{W_0}{C_L\rho s}-v^2\right)}{\frac{W_0}{C_L\rho s}-v_L{}^2}\right\} \tag{37}$$

The time t_c taken by the aircraft to come to rest and the corresponding distance x_c traversed are respectively obtained from equation(34) and (37):

$$Be(2gt\mu/\sqrt{\frac{2W_0}{C_L\rho s}}) = 1$$

Or, $t_c = \frac{1}{2g\mu}\sqrt{\frac{2W_0}{C_L\rho s}}\log\frac{1}{B} = \frac{1}{2g\mu}\sqrt{\frac{2W_0}{C_L\rho s}}\log\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}$ $\quad \frac{1}{B} > 1$ (38) Or $x_c = \frac{W_0}{C_L\rho s}.\frac{1}{g\mu}$ $\log\frac{\frac{W_0}{C_L\rho s}}{\frac{W_0}{C_L\rho s}-{v_L}^2}$ (39)

EVALUATION OF FUEL CONSUMPTI0N

To compute the fuel consumption during landing run with the thrust held equal to the drag we need to rewrite equation (2) with (34) so that

$$\frac{dW}{dt} = -cD = -\frac{c}{2}C_D\rho Sv^2 \quad (40.1)$$

$$\text{Or, } \frac{dW}{dt} = -\frac{c}{2}C_D\rho S[(\sqrt{2\frac{W_0}{C_L\rho s}})\{1-\frac{2\,B\,e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}}}{1+B\,e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}}}\}]^2 \quad (40.2)$$

which is integrated using the initial conditions (4) to find fuel spent in time t. Letting , $z = 1 + B\,e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}}$ (41)

equation (40) reduces to the form:

$$\frac{dW}{dz}\frac{dz}{dt} = -c\frac{C_D}{C_L}W_0(\frac{2-z}{z})^2 \quad \text{and} \quad \frac{dz}{dt}=B\frac{2g\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}} = \frac{2g\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}(z-1)$$

$$\frac{dW}{dz} = -c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{2g\mu}(\frac{2-z}{z})^2\frac{1}{(z-1)} = -c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{2g\mu}\{\frac{z^2-4(z-1)}{z^2(z-1)}\}$$

(From (40) 1< 1+B≤z)

$$\text{Or, } \frac{dW}{dz} = -c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{2g\mu}\{\frac{1}{(z-1)}-\frac{4}{z^2}\} < 0 \quad (42)$$

which obviously implies that as z increases ,ie, time elapses, weight of the aircraft decreases because of fuel consumption.

Initially at t=0, from (41) z=1+B, W=W_0 .Hence Solution to(42) yields

$$W= W_0 - c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{2g\mu}\{ 4(\frac{1}{z}-\frac{1}{1+B}) + \log\frac{z-1}{B}\} \qquad \{43\}$$

$$W_0 - W = c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{\frac{2B(1-e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}})}{(1+B\,e^{\frac{2gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}})(1+B)} + \frac{gt\mu}{\sqrt{\frac{2W_0}{C_L\rho s}}}\} \text{ [By use of (41)]} \qquad (44)$$

which gives the fuel consumption at any instant of time during run on the runway. Replacing t by t_c in equation (44) given by equation (38) one gets the fuel consumption on the runway till the aircraft stops or in other words the fuel consumption during landing is

$$\in= c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{\frac{B-1}{B+1}+\frac{1}{2}\log\frac{1}{B}\} \qquad (45)$$

Substituting for B from equation (34.1) is obtained

$$\in= c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{-\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}+\frac{1}{2}\log\frac{\sqrt{2\frac{W_0}{C_L\rho s}}+v_L}{\sqrt{2\frac{W_0}{C_L\rho s}}-v_L}\} = c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{-\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}+\frac{1}{2}\log\frac{1+\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}}{1-\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}}\}$$

$$= c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{-\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}+(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}\right)^3}{3}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}\right)^5}{5}\ldots+\infty\}$$ Expanding in logarithmic series,we get

$$\in = c\frac{C_D}{C_L}W_0\frac{\sqrt{\frac{2W_0}{C_L\rho s}}}{g\mu}\{\left(\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}\right)^3}{3}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho s}}}\right)^5}{5}\ldots+\infty\right\} > 0 \qquad (46)$$

We can also evaluate the fuel consumption in terms of velocity with the help of equations (31) and (40):

$$\frac{dW}{dv} = -c.\frac{1}{2}C_D\rho Sv^2.\frac{dt}{dv} = \frac{C_D}{C_L}\frac{cW_0v^2}{g\mu\{\frac{2W_0}{C_L\rho S}-v^2)} \quad = \frac{C_D}{C_L}.\frac{cW_0\{-\left(\frac{2W_0}{C_L\rho S}-v^2\right)+\frac{2W_0}{C_L\rho S\}}}{g\mu\{\frac{2W_0}{C_L\rho S}-v^2)}$$

$$\text{Or, } \frac{dW}{dv} ==- c\frac{C_D}{C_L}.\frac{W_0}{g\mu}\{1-\frac{\frac{2W_0}{C_L\rho S}\}}{(\frac{2W_0}{C_L\rho S}-v^2)}\} \qquad (47)$$

Integrating (47) :v from v_L to v and W from W_0 toW, we get

$W - W_0$

$$= - c\frac{C_D}{C_L}\frac{W_0}{g\mu}\{v - v_L + \frac{2W_0}{C_L\rho S\}}\frac{1}{2\sqrt{\frac{2W_0}{C_L\rho S}}}\log\{\left(\frac{\sqrt{\frac{2W_0}{C_L\rho S}}-v}{\sqrt{\frac{2W_0}{C_L\rho S}}+v)}\right)(\frac{\sqrt{\frac{2W_0}{C_L\rho S}}+v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}-v_L})\} \qquad (48)$$

(Expanding in logarithmic series is obtained)

$$=- c\frac{C_D}{C_L}\frac{W_0}{g\mu}\{v - v_L + \sqrt{\frac{2W_0}{C_L\rho S}}\frac{1}{2}\{\log\left(\frac{\sqrt{\frac{2W_0}{C_L\rho S}}-v}{\sqrt{\frac{2W_0}{C_L\rho S}}+v)}\right) + \log\left(\frac{\sqrt{\frac{2W_0}{C_L\rho S}}+v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}-v_L}\right)\}$$

$$=- c\frac{C_D}{C_L}\frac{W_0}{g\mu}[v - v_L + \sqrt{\frac{2W_0}{C_L\rho S}}.\frac{1}{2}\left\{\log\left(1-\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right) - \log\left(1+\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)\right\}$$

$$+\sqrt{\frac{2W_0}{C_L\rho S}}.\frac{1}{2}\{\log\left(1+\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right) - \log\left(1-\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)\}]$$

Hence the fuel consumption at any instant of time of its travel on the runway is given by

$$W_0 - W = c\frac{C_D}{C_L}\frac{W_0}{g\mu}[v - v_L - \sqrt{\frac{2W_0}{C_L\rho S}}\{(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}} + \frac{(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}})^3}{3} + \frac{(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}})^5}{5} + \ldots + \text{infinity}\}$$

$$+\sqrt{\frac{2W_0}{C_L\rho S}}\{\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}} + \frac{(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}})^3}{3} + \frac{(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}})^5}{5} + \ldots + \text{infinity}\}]$$

$$= c\frac{C_D}{C_L}\frac{W_0}{g\mu}\{-\sqrt{\frac{2W_0}{C_L\rho S}}(\frac{(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}})^3}{3} + \frac{(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}})^5}{5} + \ldots + \text{infinity})$$

$$+\sqrt{\frac{2W_0}{C_L\rho S}}\left(\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^3}{3}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^5}{5}+\ldots\ldots+\text{infinity}\right)\}$$

$$\text{Or, } W_0-W=c\frac{C_D}{C_L}\frac{W_0}{g\mu}\sqrt{\frac{2W_0}{C_L\rho S}}\{-\left(\frac{\left(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^3}{3}+\frac{\left(\frac{v}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^5}{5}+\ldots+\text{infinity}\right)$$

$$+\left(\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^3}{3}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^5}{5}+\ldots\ldots+\text{infinity}\right)\} \qquad (48)$$

gives the fuel consumption at any instant of time t or in other words when it acquires velocity v during its movement on the runway. How ever, the fuel

consumption ∈ on the runway till it stops is obtained by putting v=0 in the above equation:

$$\in= c\frac{C_D}{C_L}\frac{W_0}{g\mu}\sqrt{\frac{2W_0}{C_L\rho S}}\{\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^3}{3}+\frac{\left(\frac{v_L}{\sqrt{\frac{2W_0}{C_L\rho S}}}\right)^5}{5}+\ldots\ldots+\text{infinity})\}>0 \qquad (49)$$

After all, it is ratified that (46) and (49) represent the same equation.

CONCLUSION

Study of aircraft motion on the runway for take off and landing is significant for its own sake. During landing performance, however small may be the fuel consumption, the same is taken into consideration in order to determine velocity, distance covered on the runway as well as duration of landing. The rate of fuel consumption with constant /reversal constant thrust is uniform. The foregoing analysis indicates that greater is the touchdown velocity, greater is the fuel consumption for travel on the runway. Since the aircraft has to come to rest after completion of its journey on the runway with any of constant thrust ,constant reversal thrust and thrust equal to drag, retarded motion of the vehicle ,caused by atmospheric drag and runway road friction takes place.

REFERENCES

1.Angelo Miele(1962),Flight Mechanics, Volume 1.Theory Of Flight Paths, Addison-Wesley Publishing Company ,Inc, Reading, Massachusetts,PP 274-282.

2.Kettle,D.J.,Ground Performance At Take Off And Landing,Aircraft Engineering,Volume 30,No.347,1958.