Qualitative Analysis of Solutions of Non-linear Fractional Integrodifferential Equation

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Abstract

In this paper, we study the qualitative analysis of solutions of initial value problem of fractional order subjected to non-local conditions, by using Leray-Schauder alternative and the Bihari's integral inequality. The fractional derivatives are described in the Caputo sense, also we obtain the solutions of the integrodifferential equations in caputo sense. Moreover, example demonstrate the validity of the obtained main result.

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1 Introduction

The problem of existence and uniqueness of solution of differential equations of fractional order have been discussed by some authors which can be found in [1, 2, 3]. The purpose of this paper is to discuss the qualitative analysis of solution of the fractional integrodifferential of the form as follows:

$$\xi^{\alpha}(\mu) + \mathcal{A}\xi(\mu) = \psi(\mu, \xi(\mu), \int_0^{\mu} K(\mu, s, \xi(s))ds, \xi(\mu - 1)) \tag{1.1}$$

for $\mu \in J = [0, b], (b > 0)$ subjected to the conditions

$$\xi(\mu - 1) = \phi(\mu) \quad (0 \leq \mu < 1), \tag{1.2}$$

$$\xi(0) + g(\xi) = \xi_0 \tag{1.3}$$

where $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(\mu)$ in $\mathcal{X}$, $\psi \in C(J \times \mathcal{X} \times \mathcal{X}, \mathcal{X})$, $g \in C(C(J, \mathcal{X}), \mathcal{X})$and $\phi(\mu)$ is a continuous function for $0 \leq \mu < 1$, $\lim_{\mu \to 1-0} \phi(\mu)$ exists, for which we denote by $\phi(1 - 0) = c_0$. If we consider the solutions of (1.1) for $\mu \in J$ we get a function $\xi(\mu - 1)$ which is unable to define as solution for $0 \leq \mu < 1$. Hence, we impose some condition and the problem is reduced to fractional integrodifferential equation

$$\xi^\alpha(\mu) + \mathcal{A}\xi(\mu) = \psi(\mu, \xi(\mu), \int_0^\mu K(\mu, s, \xi(s))ds, \phi(\mu))$$

with initial conditions $\xi(0) + g(\xi) = \xi_0$. Here, it is necessary to obtain the solutions of (1.1) -(1.3) for $0 \leq \mu < b$, so that, we assume in the sequel b is not less than 1.
For the shake of simplysity let

$$B(\mu) = \int_0^\mu K(\mu, s, \xi(s))ds$$

Definition 1.1 Riemann–Liouville definition *[4, 5, 6, 7]: For $\alpha \in [n-1, n)$ the α - derivative of f is*

$$D_a^\alpha f(\xi) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n \xi}{d\mu^n} \int_a^\alpha \frac{f(\xi)}{(\mu - \xi)^{\alpha - n + 1}} d\xi$$

Definition 1.2 Caputo definition*[4, 5, 6, 7]:For $\alpha \in (n-1, n)$ the α - derivative of f is*

$${}_a^C D_\mu^\alpha f(\mu) = \frac{1}{\Gamma(\alpha - n)} \int_a^\mu \frac{f^{(n)}(\tau)}{(\mu - \tau)^{\alpha - n + 1}} d\tau$$

2 Non-linear Fractional Integrodifferential Equation with Non-local Condition

2.1 Preliminaries

Definition 2.1 *Let $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $T(\mu), \mu\mu \geq 0$, on a Banach space $\mathcal{X}$. The function $\xi \in B$ given by*

$$\xi(\mu) = T(\mu)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha - n)} \int_0^\mu \frac{T(\mu - s)\psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}} \tag{2.1.1}$$

for $0 \leq \mu < 1$, and

$$\xi(\mu) = T(\mu)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha - n)} \int_0^\mu \frac{T(\mu - s)\psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}}$$
$$+ \frac{1}{\Gamma(-\alpha - n)} \int_1^\mu \frac{T(\mu - s)\psi^{(n)}(s, \xi(s), B(s), \xi(s-1))ds}{(\mu - s)^{-\alpha+1-n}} \tag{2.1.2}$$

for $1 \leq \mu \leq b$, is called the mild solution of the problem (1.1) -(1.3).

Lemma 2.2 Leray–Schauder alternative*[8]: Let $\Phi : E \to E$ be a completely continuous operator. Let $\mu(\Phi) = \{\xi \in E : \xi = \lambda\Phi(\xi) for some, 0 < \lambda < 1\}$. then, either the set $\mu(\Phi)$ is unbounded, or Φ has at least one fixed point.*

Lemma 2.3 *[9] Let $u(\mu), p(\mu) \in C(R_+, R_+), R_+ = [0, \infty)$. let $w(u)$ be a continuous, non-decreasing function defined on $R_+, w(u) > 0$ for $u > 0$ and $w(0) = 0$. if*

$$u(\mu) \leq c + \frac{1}{\Gamma(-\alpha - n)} \int_0^\mu \frac{p^{(n)}(s)w(u(s))ds}{(\mu - s)^{-\alpha+1-n}}$$

for $\mu \in R_+$, where $c \geq 0$ is a constant, then for $0 \leq \mu \leq \mu_1$,

$$u(\mu) \leq W^{-1}\left[W(c) + \frac{1}{\Gamma(-\alpha - n)} \int_0^\mu \frac{p^{(n)}(s)ds}{(\mu - s)^{-\alpha+1-n}}\right],$$

where

$$W(r) = \int_{r_0}^r \frac{ds}{w(s)} \frac{1}{\zeta^*}, \quad r > 0, r_0 > 0,$$

W^{-1} is the inverse function of W and $T_1 \in R_+$ be chosen so that

$$W(c) + \frac{1}{\Gamma(-\alpha - n)} \int_0^\mu \frac{p^{(n)}(s)ds}{(\mu - s)^{-\alpha+1-n}} \in Dom(W^{-1}),$$

for all $\mu \in R_+$ lying in the interval $0 \leq \mu \leq \mu_1$.

We consider the following hypotheses for further use:
(H_1) If $\mathcal{A}$ is the infinitesimal generator of a semigroup of bounded linear operators $T(\mu)$ in $\mathcal{X}$, which is compact for $\mu > 0$, then $\exists$ a constant $M \geq 1$ such that $\| T(\mu) \| \leq M$, $\mu \geq 0$.
(H_2) if $\xi, y, z \in \mathcal{X}$, and $p^{(n)} \in C(J, R_+)$,then The function ψ in (1.1) satisfies the condition

$$\| \psi^n(\mu, \xi, y, z) \| \leq p^{(n)}(\mu) \left[\Omega\left(\| \xi \| \right) + \Omega\left(\| y \| \right) + \Omega\left(\| z \| \right) \right],$$

and $\Omega : R_+ \to (0, \infty)$ is continuous and increasing function satisfying $\Omega\left(\alpha(\mu) \| \xi \|\right) \leq \alpha(\mu)\Omega\left(\| \xi \| \right)$,where α is defined as the function p.
(H_3) If $\mu \geq s \geq 0$ and $\xi \in \mathcal{X}$,then There $\exists$ a continuous function $q : J \to R_+$ such that

$$\| \int_0^\mu K(\mu, s, \xi(s)) ds \| \leq q(\mu) \| \xi \|$$

(H_4) If $\xi \in B$,then $\exists$ constant $G > 0$ such that

$$\| g(\xi) \| \leq G$$

(H_5) If $\mu \in J$,then for every $(\xi, y, z) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$, the function $\psi(, \xi, y, z) : J \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and the function $\psi(\mu, ., ., .) : J \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are strongly measurable and continuous respectively.
(H_6) If $\mu, s \in J$, then for every $\xi \in \mathcal{X}$, the function $k(., ., \xi) : J \times J \times \mathcal{X} \to \mathcal{X}$ and the function $k(\mu, s, .) : J \times J \times \mathcal{X} \to \mathcal{X}$ are strongly measurable and continuous respectively.
(H_7) If m is positive integer ,then $\exists$ $\alpha_m \in L^1(J)$ such that

$$\sup_{\|\xi\| \leq m, \|y\| \leq m, \|z\| \leq m} \| \psi^n(\mu, \xi, y, z) \| \leq \alpha_m(\mu) \ for \ \mu \in J \ a.e.$$

(H_8)If $\xi, y, z, \overline{\xi}, \overline{y}, \overline{z} \in X$, and $\overline{p^{(n)}} \in C(R_+, R_+)$,then The function ψ in (1.1) satisfies the condition

$$\| \psi^n(\mu, \xi, y, z) - \psi^n(\mu, \overline{\xi}, \overline{z}, \overline{z}) \| \leq p^{(n)}(\mu) \left[\Omega\left(\| \xi - \overline{\xi} \| \right) + \Omega\left(\| y - \overline{y} \| \right) + \Omega\left(\| z - \overline{z} \| \right) \right],$$

and $\overline{\Omega}(u)$ is a continuous and increasing function for $u \geq 0, \overline{\Omega}(0)$.
(H_9)If $\xi, \overline{\xi}, \in \mathcal{X}$, and $\overline{p^{(n)}} \in C(R_+, R_+)$,then The function k in (1.4) satisfies the condition

$$\| \int_0^\mu \left[K(\mu, s, \xi) - K(\mu, s, \overline{\xi}) \right] ds \| \leq \overline{q}(\mu) \| \xi - \overline{\xi} \|,$$

(H_{10}) There exist constant $\overline{G} > 0$ such that

$$\| g(\xi) - g(\overline{\xi}) \| \leq \overline{G} \| \xi - \overline{\xi} \| \ \ for \ every \ \xi, \overline{\xi}, \in B$$

and $M\overline{G} < 1$.

2.2 Main Result

Theorem 2.4 *If $(H_1) - (H_7)$ hold. Then (1.1) -(1.3) has a solution $\xi(\mu)$ defined on J provided b satisfies*

$$\frac{1}{\Gamma(-\alpha-n)}\int_0^b \frac{M[p^{(n)}(s)(1+q(s))+p^{(n)}(s+1)]ds}{(\mu-s)^{-\alpha+1-n}} < \frac{1}{\Gamma(-\alpha-n)}\int_c^\infty \frac{ds}{\Omega(s)(\mu-s)^{-\alpha+1-n}}, \tag{2.2.1}$$

where

$$c = M\left[\| \xi_0 \| + G\right] + \frac{M}{\Gamma(-\alpha-n)}\int_0^1 \frac{p^{(n)}(s)\Omega\left(\| \phi(s) \|\right)ds}{(\mu-s)^{-\alpha+1-n}} \tag{2.2.2}$$

Proof. An operator $\Phi : B \to B$ is defined by

$$(\Phi\xi)(\mu) = T(\mu)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha-n)}\int_0^\mu \frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}} \tag{2.2.3}$$

for $0 \leq \mu < 1$, and

$$(\Phi\xi)(\mu) = T(c)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha-n)}\int_0^1 \frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$+ \frac{1}{\Gamma(-\alpha-n)}\int_1^\mu \frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\xi(s-1))ds}{(\mu-s)^{-\alpha+1-n}} \tag{2.2.4}$$

for $1 \leq \mu \leq b$,
instead to use (2.1.2), we establish the priori bounds on the solutions of the problem

$$\xi^\alpha(\mu) + \mathcal{A}\xi(\mu) = \lambda\psi(\mu,\xi(\mu\mu),B(\mu),\xi(\mu-1)) \tag{2.2.5}$$

under the initial conditions (1.2) -(1.3)for $\lambda \in (0,1)$. Let $\xi(\mu)$ be a solution of (2.2.5) with (1.2)-(1.3), then we consider the following two cases.
Case I: If $0 \leq \mu \leq 1$, then From the hypotheses, we have

$$\| \xi(\mu) \| = \| T(\mu)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha-n)}\int_0^\mu \frac{\lambda T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}} \|$$

$$\leq \| T(\mu) \| \| [\xi_0 - g(\xi)] \| + \int_0^\mu \frac{Mp^{(n)}(s)\left[\Omega\left(\| \xi(s) \|\right) + \Omega\left(\| B(s) \|\right) + \Omega\left(\| \phi(s) \|\right)\right]ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$\leq M\left[\| \xi_0 \| + G\right] + \int_0^t \frac{Mp^{(n)}(s)\Omega\left(\| \phi(s) \|\right)ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|} + \int_0^\mu \frac{Mp^{(n)}(s)\left[\Omega\left(\| \xi(s) \|\right) + \Omega\left(q(s) \| \xi(s) \|\right)\right]ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$\leq M\left[\| \xi_0 \| + G\right] + \int_0^1 \frac{Mp^{(n)}(s)\Omega\left(\| \phi(s) \|\right)ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|} + \int_0^\mu \frac{Mp^{(n)}(s)\left[\Omega\left(\| \xi(s) \|\right) + q(s)\Omega\left(\| \xi(s) \|\right)\right]ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$= c + \int_0^\mu \frac{Mp^{(n)}(s)\,[1+q(s)]\,\Omega\,(\| \xi(s) \|)\,ds}{\zeta^*} \tag{2.2.6}$$

where $\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} = \zeta$ and $\| \zeta \| = \zeta^*$
Let $u(\mu)$be defined by the right hand side of (2.2.6), then $u(0 = c$, $\| \xi(\mu) \| \leq u(\mu) \|$ and

$$u^\alpha(\mu) = Mp^{(n)}(\mu)\,[1+q(\mu)]\,\Omega\,(\| x(\mu) \|) \leq Mp^{(n)}(\mu)\,[1+q(\mu)]\,\Omega(u(\mu));$$

that is ,

$$\frac{u^\alpha(\mu)}{\Omega(u(\mu))} \leq Mp^{(n)}(\mu)\,[1+q(\mu)]\,\Omega(u(\mu)) \tag{2.2.7}$$

Integration of (2.2.7) from 0 to μ $(0 \leq \mu < 1)$, the change of variable $\mu \to s = u(\mu)$, and the condition (2.1.1) gives

$$\int_c^{u(\mu)} \frac{ds}{\Omega(s)\zeta^*} \leq \int_0^\mu \frac{Mp^{(n)}(s)\,[1+q(s)]\,ds}{\zeta^*} \leq \int_0^1 \frac{Mp^{(n)}(s)\,[1+q(s)]\,ds}{\zeta^*} < \int_c^\infty \frac{ds}{\Omega(s)\zeta^*} \tag{2.2.8}$$

By this inequality and the mean value theorem, we conclude that, there is a constant γ_1 independent of $\lambda \in (0,1)$ such that $u(\mu) \leq \gamma_1$ for $0 \leq \mu < 1$ and hence $\| \xi(\mu) \| \leq \gamma_1$.
Case II: If $0 \leq \mu \leq b$, then From the hypotheses, we have

$$\| \xi(\mu) \| = \| T(\mu)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha-n)} \int_0^1 \frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$+ \frac{1}{\Gamma(-\alpha-n)} \int_1^\mu \frac{\lambda T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),x(s-1))ds}{(\mu-s)^{-\alpha+1-n}} \|$$

$$\leq \| T(\mu) \| \| [\xi_0 - g(\xi)] \| + \int_0^1 \frac{Mp^{(n)}(s)\,[\Omega\,(\| \xi(s) \|) + \Omega\,(\| B(s) \|) + \Omega\,(\| \phi(s) \|)]\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$+ \int_0^\mu \frac{Mp^{(n)}(s)\,[\Omega\,(\| \xi(s) \|) + \Omega\,(\| B(s) \|) + \Omega\,(\| \xi(s-1) \|)]\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$\leq M\,[\| \xi_0 \| + G] + \int_0^1 \frac{Mp^{(n)}(s)\Omega\,(\| \phi(s) \|)\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|} + \int_0^1 \frac{Mp^{(n)}(s)\,[1+q(s)]\,\Omega\,(\| x(s) \|)\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$+ \int_1^\mu \frac{Mp^{(n)}(s)\,[1+q(s)]\,\Omega\,(\| \xi(s) \|)\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|} + \int_1^\mu \frac{Mp^{(n)}(s)\Omega\,(\| \xi(s-1) \|)\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$= c + \int_0^\mu \frac{Mp^{(n)}(s)\,[1+q(s)]\,\Omega\,(\| \xi(s) \|)\,ds}{\zeta^*} + I_1 \tag{2.2.9}$$

where

$$I_1 = \int_1^\mu \frac{Mp^{(n)}(s)\Omega\,(\| \xi(s-1) \|)\,ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|} \tag{2.2.10}$$

By changing the variable, from (2.2.10), we get

$$I_1 = \int_0^{\mu-1} \frac{Mp^{(n)}(\sigma+1)\Omega\left(\| \xi(\sigma) \|\right) d\sigma}{\zeta^*} \leq \int_0^t \frac{Mp^{(n)}(\sigma+1)\Omega\left(\| \xi(\sigma) \|\right) d\sigma}{\zeta^*} \tag{2.2.11}$$

Using this inequality in (2.2.9), we get

$$\| \xi(\mu) \| \leq c + \frac{1}{\Gamma(-\alpha-n)} \int_0^\mu \frac{M[p^{(n)}(s)(1+q(s)) + p^{(n)}(s+1)]\Omega\left(\| \xi(s) \|\right) ds}{(\mu-s)^{-\alpha+1-n}} \tag{2.2.12}$$

Let $v(\mu)$ be defined by the right hand side of (2.2.12), then $v(0) = c$, $\| \xi(\mu) \| \leq v(t)$ and

$$v^\alpha(\mu) = M\left[p^{(n)}(\mu)(1+q(t)) + p^{(n)}(\mu+1)\right]\Omega\left(\| \xi(\mu) \|\right)$$

$$\leq M\left[p^{(n)}(\mu)(1+q(\mu)) + p^{(n)}(\mu+1)\right]\Omega\left(v(\mu)\right);$$

that is,

$$\frac{v^\alpha(\mu)}{\Omega\left(v(\mu)\right)} \leq M\left[p^{(n)}(\mu)(1+q(\mu)) + p^{(n)}(\mu+1)\right]. \tag{2.2.13}$$

Integration of (2.2.13) from 0 to μ, $1 \leq \mu \leq b$, the change of variable, and the condition (2.1.1) gives

$$\int_c^{v(\mu)} \frac{ds}{\Omega(s)\zeta^*} \leq \frac{1}{\Gamma(-\alpha-n)} \int_0^\mu \frac{M[p^{(n)}(s)(1+q(s)) + p^{(n)}(s+1)]ds}{(\mu-s)^{-\alpha+1-n}}$$

$$\leq \frac{1}{\Gamma(-\alpha-n)} \int_0^b \frac{M[p^{(n)}(s)(1+q(s)) + p^{(n)}(s+1)]ds}{(\mu-s)^{-\alpha+1-n}} < \int_c^\infty \frac{ds}{\Omega(s)\zeta^*} \tag{2.2.14}$$

By(2.2.14) we conclude that there is a constant γ_2 independent of $\lambda \in (0,1)$ such that $v(\mu) \leq \gamma_2$ and hence $\| x(\mu) \| \leq \gamma_2$ for $1 \leq \mu \leq b$. Let $\gamma = max\{\gamma_1, \gamma_2\}$. Obviously, $\| \xi(\mu) \| \leq \gamma$ for $\mu \in J$ and consequently, $\| \xi \| = sup\{\| \xi(\mu) \| : \mu \in J\} \leq \gamma$.

Claim 1: Φ is completely continuous.

Let $B_m = \{\xi \in B : \| \xi(\mu) \| \leq m, \mu \in J\}$, for some $m \geq 1$. Then for each $m \geq 1$, the set B_m is clearly closed, convex and bounded subset of B. First we show that ΦB_m is uniformly bounded. We have to consider the two cases.

Case I. If $0 \leq \mu \leq 1$, then From the definition of the operator Φ as in (2.2.3) ,hypotheses and the fact that $\xi \in B_m$, we obtain

$$\| (\Phi\xi)(\mu) \| \leq \| T(\mu) \| [\| \xi_0 \| + G] + \int_0^\mu \frac{\| T(\mu-s)\psi^{(n)}(s, \xi(s), B(s), \phi(s)) \| ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$\leq M[\| \xi_0 \| + G] + \int_0^1 M\alpha_m(s)ds\frac{1}{\zeta^*}$$

$$\leq M[\| \xi_0 \| + G] + M \| \alpha_m \| L^1\frac{1}{\zeta^*} \tag{2.2.15}$$

Case II. If $0 \leq \mu \leq b$, then From (2.2.4), hypotheses and the fact that $\xi \in B_m$, then looking at
Case I. immediately we have

$$\| (F\xi)(\mu) \| \leq \| T(\mu) \| [\| \xi_0 \| + G] + \int_0^1 \frac{M \| \psi^{(n)}(s, \xi(s), B(s), \phi(s)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$+ \int_1^\mu \frac{M \| \psi^{(n)}(s, \xi(s), B(s), \xi(s-1)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$\leq M[\| \xi_0 \| + G] + \int_0^1 M\alpha_m(s)ds\frac{1}{\zeta^*} + \int_1^\mu M\alpha_m(s)ds\frac{1}{\zeta^*}$$

$$= M[\| \xi_0 \| + G] + \int_0^\mu M\alpha_m(s)ds\frac{1}{\zeta^*}$$

$$\leq M[\| \xi_0 \| + G] + M \| \alpha_m \| L^1\frac{1}{\zeta^*} \tag{2.2.16}$$

From (2.2.15) and (2.2.16), it follows that $\{\Phi B_m\}$ is uniformly bounded.
Sub claim 1: Φ maps B_m into an equicontinuous family.
Let $\xi \in B_m$. We must consider three cases.
Case I. If μ_1 and μ_2 are contained in $0 \leq \mu < 1$, then From (2.2.3), it follows that

$$(\Phi\xi)(\mu_1) - (\Phi\xi)(\mu_2) = T(\mu_1)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\mu_1} \frac{T(\mu_1 - s)\psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}}$$

$$-T(\mu_2)[\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\mu_2} \frac{T(\mu_2 - s)\psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}}$$

$$= [T(\mu_1) - T(\mu_2)] [\xi_0 - g(\xi)] + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\mu_1} \frac{[T(\mu_1 - s) - T(\mu_2 - s)] \psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}}$$

$$+ \frac{1}{\Gamma(-\alpha - n)} \int_{\mu_1}^{\mu_2} \frac{T(\mu_2 - s)\psi^{(n)}(s, \xi(s), B(s), \phi(s))ds}{(\mu - s)^{-\alpha+1-n}} \tag{2.2.17}$$

By the above equality and hypotheses, we have

$$\| (\Phi\xi)(\mu_1) - (\Phi\xi)(\mu_2) \| \leq \| T(\mu_1) - T(\mu_2) \| [\| \xi_0 \| + G]$$

$$+ \int_0^{\mu_1} \frac{\| T(\mu_1 - s) - T(\mu_2 - s) \| \| \psi^{(n)}(s, \xi(s), B(s), \phi(s)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$+\int_{\mu_1}^{\mu_2} \frac{\| T(\mu_2 - s) \| \| \psi^{(n)}(s, \xi(s), B(s), \phi(s)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$\leq \| T(\mu_1) - T(\mu_2) \| [\| \xi_0 \| + G] + \int_0^{\mu_1} \| T(\mu_1 - s) - T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*}$$

$$+\int_{\mu_1}^{\mu_2} \| T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*} \tag{2.2.18}$$

Case II. If μ_1 and μ_2 are contained in $0 \leq \mu < b$, then From (2.2.4), it follows that

$$(\Phi\xi)(\mu_1) - (\Phi\xi)(\mu_2) = [T(\mu_1) - T(\mu_2)] [\xi_0 - g(\xi)]$$

$$+\frac{1}{\Gamma(-\alpha - n)} \int_0^1 \frac{[T(\mu_1 - s) - T(\mu_2 - s)] \psi^{(n)}(s, \xi(s), B(s), \phi(s)) ds}{(\mu - s)^{-\alpha+1-n}}$$

$$+\frac{1}{\Gamma(-\alpha - n)} \int_1^{\mu_1} \frac{[T(\mu_1 - s) - T(\mu_2 - s)] \psi^{(n)}(s, \xi(s), B(s), \xi(s-1)) ds}{(\mu - s)^{-\alpha+1-n}}$$

$$+\frac{1}{\Gamma(-\alpha - n)} \int_{\mu_1}^{\mu_2} \frac{T(\mu_2 - s) \psi^{(n)}(s, \xi(s), B(s), \xi(s-1)) ds}{(\mu - s)^{-\alpha+1-n}} \tag{2.2.19}$$

By the above equality and hypotheses, we have

$$\| (\Phi\xi)(\mu_1) - (\Phi\xi)(\mu_2) \| \leq \| T(\mu_1) - T(\mu_2) \| [\| \xi_0 \| + G]$$

$$+\int_0^1 \frac{\| T(\mu_1 - s) - T(\mu_2 - s) \| \| \psi^{(n)}(s, \xi(s), B(s), \phi(s)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$+\int_1^{\mu_1} \frac{\| T(\mu_1 - s) - T(\mu_2 - s) \| \| \psi^{(n)}(s, \xi(s), B(s), \xi(s-1)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$+\int_{\mu_1}^{\mu_2} \frac{\| T(\mu_2 - s) \| \| \psi^{(n)}(s, \xi(s), B(s), \xi(s-1)) \| ds}{\| \Gamma(-\alpha - n)(\mu - s)^{-\alpha+1-n} \|}$$

$$\leq \| T(\mu_1) - T(\mu_2) \| [\| \xi_0 \| + G] + \int_0^1 \| T(\mu_1 - s) - T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*}$$

$$+\int_1^{t_1} \| T(\mu_1 - s) - T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*}$$

$$+\int_{\mu_1}^{\mu_2} \| T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*}$$

$$\leq \| T(\mu_1) - T(\mu_2) \| [\| \xi_0 \| + G] + \int_0^{t_1} \| T(\mu_1 - s) - T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*}$$

$$+\int_{\mu_1}^{\mu_2} \| T(\mu_2 - s) \| \alpha_m(s) ds \frac{1}{\zeta^*} \tag{2.2.20}$$

Case III. If μ_1 and μ_2 are respectively contained in $[0,1)$ and $[1,b]$, then From (2.2.3)and (2.2.4), it follows that

$$(\Phi\xi)(\mu_1)-(\Phi\xi)(\mu_2) = T(\mu_1)(\xi_0-g(\xi))+\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu_1}\frac{T(\mu_1-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-T(\mu_2)(\xi_0-g(\xi))-\frac{1}{\Gamma(-\alpha-n)}\int_0^{1}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-\frac{1}{\Gamma(-\alpha-n)}\int_1^{\mu_2}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\xi(s-1)))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$=[T(\mu_1)-T(\mu_2)]\,[\xi_0-g(\xi)]+\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu_1}\frac{T(\mu_1-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu_1}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-\frac{1}{\Gamma(-\alpha-n)}\int_{\mu_1}^{1}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-\frac{1}{\Gamma(-\alpha-n)}\int_1^{\mu_2}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\xi(s-1)))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$=[T(\mu_1)-T(\mu_2)]\,[\xi_0-g(\xi)]+\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu_1}\frac{[T(\mu_1-s)-T(\mu_2-s)]\,\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

$$-\left[\int_{\mu_1}^{1}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}+\int_1^{\mu_2}\frac{T(\mu_2-s)\psi^{(n)}(s,\xi(s),B(s),\xi(s-1)))ds}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right] \tag{2.2.21}$$

By the above equality and hypotheses, we have

$$\|(\Phi\xi)(\mu_1)-(\Phi\xi)(\mu_2)\|\leq\|T(\mu_1)-T(\mu_2)\|\,[\|\xi_0\|+G]$$

$$+\int_0^{\mu_1}\frac{\|T(\mu_1-s)-T(\mu_2-s)\|\|\psi^{(n)}(s,\xi(s),B(s),\phi(s))\|ds}{\|\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}\|}$$

$$\int_{\mu_1}^{1}\frac{\|T(\mu_2-s)\|\|\psi^{(n)}(s,\xi(s),B(s),\phi(s))\|ds}{\|\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}\|}$$

$$\int_1^{\mu_2}\frac{\|T(\mu_2-s)\|\|\psi^{(n)}(s,\xi(s),B(s),\xi(s-1)))\|ds}{\|\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}\|}$$

$$\leq\|T(\mu_1)-T(\mu_2)\|\,[\|\xi_0\|+G]+\int_0^{\mu_1}\|T(\mu_1-s)-T(\mu_2-s)\|\,\alpha_m(s)ds\frac{1}{\zeta^*}$$

$$+\int_{\mu_1}^{1}\|T(\mu_2-s)\|\,\alpha_m(s)ds\frac{1}{\zeta^*}+\int_1^{\mu_2}\|T(\mu_2-s)\|\,\alpha_m(s)ds\frac{1}{\zeta^*}$$

$$=\| T(\mu_1)-T(\mu_2) \| [\| \xi_0 \| +G]+\int_0^{\mu_1} \| T(\mu_1-s)-T(\mu_2-s) \| \alpha_m(s)ds\frac{1}{\zeta^*}$$
$$+\int_{\mu_1}^{\mu_2} \| T(\mu_2-s) \| \alpha_m(s)ds\frac{1}{\zeta^*} \qquad (2.2.22)$$

From (2.2.18), (2.2.20), (2.2.22), the right side of each one is independent of $\xi \in B_m$ and tends to zero as $\mu_1-\mu_2 \to 0$; since the compactness of $T(\mu)$ for $\mu>0$ implies the continuity in the uniform operator topology. Thus Φ maps B_m into an equicontinuous family of functions and uniformly bounded.

Claim 2: $\overline{\Phi B_m}$ is compact.

Since we have shown that ΦB_m is an equicontinuous and uniformly bounded collection, it is sufficient to show that Φ maps B_m into a pre compact set in $\mathcal{X}$. We have to consider the following two cases.

Case I: If $0<\mu<1$ is fixed and ϵ a real number satisfying $0<\epsilon<1$, then for $\xi \in B_m$ we define

$$(\Phi_\epsilon\xi)(\mu)=T(\mu)[\xi_0-g(\xi)]+\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu-\epsilon}\frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$
$$=T(\mu)[\xi_0-g(\xi)]+T(\epsilon)\frac{1}{\Gamma(-\alpha-n)}\int_0^{\mu-\epsilon}\frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$

Since $T(\mu)$ is a compact operator, the set $Y_\epsilon(\mu)=\{(\Phi_\epsilon\xi)(\mu):\xi\in B_m\}$ is pre compact in $\mathcal{X}$ $\forall\epsilon$, $0<\epsilon<\mu$ and $0<\mu<1$. Moreover, $\forall\xi\in B_m$, we get

$$\| (\Phi\xi)(\mu)-(\Phi_\epsilon\xi)(\mu) \| +\int_{\mu-\epsilon}^{\mu}\frac{\| T(\mu-s) \| \| \psi^{(n)}(s,\xi(s),B(s),\phi(s)) \| ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$
$$\leq M\int_{\mu-\epsilon}^{\mu}\alpha_m(s)\frac{ds}{\zeta^*} \qquad (2.2.23)$$

Therefore there are pre compact sets arbitrary close to the set $\{(\Phi\xi)(\mu):\xi\in B_m\}$ for $0<\mu<1$. Hence the set $\{(\Phi\xi)(\mu):\xi\in B_m\}$ is pre compact in $\mathcal{X}$ for $0<\mu<1$.

Case II: If $0<\mu<b$ is fixed and ϵ a real number satisfying $0<\epsilon<1$, then for $\xi\in B_m$ we define

$$(\Phi_\epsilon\xi)(\mu)=T(\mu)[\xi_0-g(\xi)]+\frac{1}{\Gamma(-\alpha-n)}\int_0^{1}\frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\phi(s))ds}{(\mu-s)^{-\alpha+1-n}}$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_1^{\mu-\epsilon}\frac{T(\mu-s)\psi^{(n)}(s,\xi(s),B(s),\xi(s-1))ds}{(\mu-s)^{-\alpha+1-n}}$$

Since $T(\mu)$ is a compact operator, the set $\overline{Y}_\epsilon(\mu)=\{(\Phi_\epsilon\xi)(\mu):\xi\in B_m\}$ is pre compact in $\mathcal{X}$ $\forall\epsilon$, $1<\epsilon<\mu$ and $1<\mu<b$. Moreover, $\forall\xi\in B_m$, we get

$$\| (\Phi\xi)(\mu)-(\Phi_\epsilon\xi)(\mu) \| \leq \int_{\mu-\epsilon}^{\mu}\frac{\| T(\mu-s) \| \| \psi^{(n)}(s,\xi(s),B(s),s(s-1)) \| ds}{\| \Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n} \|}$$

$$\leq M\int_{\mu-\epsilon}^{\mu}\alpha_m(s)\frac{ds}{\zeta^*} \tag{2.2.24}$$

Therefore there are pre compact sets arbitrary close to the set $\{(\Phi\xi)(\mu):\xi\in B_m\}$ for $1<\mu<b$. Hence the set $\{(\Phi\xi)(\mu):\xi\in B_m\}$ is pre compact in $\mathcal{X}$ for $1<\mu<b$. On combining these two cases we conclude that the set $\{(\Phi\xi)(\mu):\xi\in B_m\}$ is pre compact in $\mathcal{X}$ for $\mu\in J$.

Sub claim 2: $\Phi: B\to B$ is continuous.

Let $\{v_n\}$ be a sequence of elements of B converging to v in B. Then there exists an integer q such that $\| v_n(\mu)\|\leq q$ for all n and $\mu\in J$, so $v_n\in B_q$ and $u\in B_q$. We consider the following two cases.

Case I: $\forall\mu\in[0,1)$ and by $(H_4)-(H_7)$, we have

$$\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))\to\psi^{(n)}(s,v(s),B(s),\phi(s)).$$

thus, since

$$\|\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))-\psi^{(n)}(s,v(s),B(s),\phi(s))\|\leq 2\alpha_q,$$

we have from dominated convergence

$$\|\Phi v_n-\Phi v\| = Sup_{\mu\in[0,1)}\| T(\mu)[g(v_n)-g(v)]$$

$$+\int_0^{\mu}T(\mu-s)\left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right]ds\,\|$$

$$\leq M[g(v_n)-g(v)]+M\int_0^{t}\left\|\left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right]ds\right\|\to 0. \tag{2.2.25}$$

Case II: If $\mu\in[1,b)$ and from $(H_4)-(H_7)$, then we have

$$\psi^{(n)}(s,v_n(s),B_n(s),v_n(s-1))\to\psi^{(n)}(s,v(s),B(s),v(s-1)).$$

thus, since

$$\|\psi^{(n)}(s,v_n(s),B_n(s),v_n(s-1))-\psi^{(n)}(s,v(s),B(s),v(s-1))\|\leq 2\alpha_q,$$

we have from dominated convergence

$$\|\Phi v_n-\Phi v\| = Sup_{\mu\in[0,1)}\| T(\mu)[g(v_n)-g(v)]$$

$$+\int_0^{1}T(\mu-s)\left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right]ds$$

$$+\int_1^{\mu}T(\mu-s)\left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),v_n(s-1))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),v(s-1))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right]ds\,\|$$

$$\leq M[g(v_n)-g(v)]+M\int_0^1 \| \left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right] ds \|$$

$$+M\int_0^\mu \| \left[\frac{\psi^{(n)}(s,v_n(s),B_n(s),v_n(s-1))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,v(s),B(s),v(s-1))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right] ds \|\to 0 \tag{2.2.26}$$

From (2.2.25) and (2.2.26), we conclude that the operator Φ is continuous. Thus Φ is completely continuous.
Finally, the set $\mu(\Phi)=\{\xi\in B: x=\lambda\Phi\xi, \lambda\in(0,1)\}$ is bounded which was proved in the first Claim. Consequently, by Lemma 2.2, the operator Φ has a fixed point in B. This implies that the problem $(1.1)-(1.3)$ has a solution exist.

Theorem 2.5 *Suppose that (H_1), $(H_8)-(H_{10})$ hold. Let*

$$\gamma(r)=\int_{r_0}^{r}\frac{ds}{\overline{\Omega}(s)}\frac{1}{\zeta^*},\ (0<r_0\leq r)$$

with γ^{-1} being the inverse function of γ and assume that $\lim_{r_0\to+0}\gamma(r)=\infty$ for any fixed r. Then (1.1) -(1.3) has at most one solution on R_+.

Proof. Let $\xi(\mu),y(\mu)$ be two solutions of equation (1.1), under the initial conditions

$$\xi(\mu-1)=y(\mu-1)=\phi(\mu),\ (0\leq t<1), \tag{2.2.27}$$

$$\xi(0)+g(\xi)=\xi_0 \textit{ and } y(0)+g(y)=\xi_0,$$

and let $u(\mu)=\|\xi(\mu)-y(\mu)\|, \mu\in R_+$. We consider the following two cases.
Case I. if $0\leq\mu<1$, then From the hypotheses, we have

$$u(\mu)\leq\| T(\mu)\|\| g(\xi)-g(y)\|$$

$$+\int_0^\mu \| T(\mu-s)\|\| \left[\frac{\psi^{(n)}(s,\xi(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}-\frac{\psi^{(n)}(s,y(s),B(s),\phi(s))}{\Gamma(-\alpha-n)(\mu-s)^{-\alpha+1-n}}\right]\| ds$$

$$\leq M\overline{G}\|\xi-y\|+\int_0^\mu M\overline{p^{(n)}}(s)\left[\overline{\Omega}\left(\|\xi(s)-y(s)\|\right)+\overline{\Omega}\left(\overline{q}(s)\|\xi(s)-y(s)\|\right)\right]ds\frac{1}{\zeta^*}$$

$$=M\overline{G}u(\mu)+\int_0^\mu M\overline{p^{(n)}}(s)(1+\overline{q}(s))\overline{\Omega}(u(s))ds\frac{1}{\zeta^*},$$

which implies

$$u(\mu)\leq\int_0^\mu\frac{M\overline{p^{(n)}}(s)(1+\overline{q}(s))}{1-M\overline{G}}\overline{\Omega}(u(s))ds\frac{1}{\zeta^*}$$

$$\leq\epsilon_1+\int_0^\mu\frac{M\overline{p^{(n)}}(s)(1+\overline{q}(s))}{1-M\overline{G}}\overline{\Omega}(u(s))ds\frac{1}{\zeta^*} \tag{2.2.28}$$

where $\epsilon_1 > 0$ is sufficiently small constant. By using Lemma 2.3 to (2.2.28) we get

$$\| \xi(\mu) - y(\mu) \| \le \gamma^{-1}\left[\gamma(\epsilon_1) + \int_0^t \frac{M\overline{p^{(n)}}(s)(1+\overline{q}(s))}{1-M\overline{G}} ds \frac{1}{\zeta^*}\right] \tag{2.2.29}$$

Case II. If $1 \le \mu < \infty$, then From the hypotheses, we have

$$\begin{aligned} u(\mu) &\le M\overline{G}u(\mu) + \int_0^1 M\overline{p^{(n)}}(s)(1+\overline{q}(s))\overline{\Omega}(u(s))ds\frac{1}{\zeta^*} + \int_1^\mu M\overline{p^{(n)}}(s)(1+\overline{q}(s))\overline{\Omega}(u(s))ds\frac{1}{\zeta^*} \\ &\quad + \int_1^\mu M\overline{p(n)}(s)\overline{\Omega}(u(s-1))ds\frac{1}{\zeta^*} \\ &= M\overline{G}u(\mu) + \int_0^\mu M\overline{p^{(n)}}(s)(1+\overline{q}(s))\overline{\Omega}(u(s))ds\frac{1}{\zeta^*} + I_2 \end{aligned} \tag{2.2.30}$$

where

$$I_2 = \int_1^\mu M\overline{p^{(n)}}(s)\overline{\Omega}(u(s-1))ds\frac{1}{\zeta^*}.$$

By the change of variable, we observe that

$$I_2 \le \int_0^\mu M\overline{p^{(n)}}(s+1)\overline{\Omega}(u(s))ds\frac{1}{\zeta^*}. \tag{2.2.31}$$

Using (2.2.31) in (2.2.30), we obtain

$$u(\mu) \le M\overline{G}u(\mu) + \int_0^\mu M\left[\overline{p^{(n)}}(s)(1+\overline{q}(s)) + \overline{p^{(n)}}(s+1)\right]\overline{\Omega}(u(s))ds\frac{1}{\zeta^*},$$

which implies

$$\begin{aligned} u(\mu) &\le \int_0^\mu \frac{M\left[\overline{p^{(n)}}(s)(1+\overline{q}(s)) + \overline{p^{(n)}}(s+1)\right]}{1-M\overline{G}}\overline{\Omega}(u(s))ds\frac{1}{\zeta^*}, \\ &\le \epsilon_2 + \int_0^t \frac{M\left[\overline{p^{(n)}}(s)(1+\overline{q}(s)) + \overline{p^{(n)}}(s+1)\right]}{1-M\overline{G}}\overline{\Omega}(u(s))ds\frac{1}{\zeta^*}, \end{aligned} \tag{2.2.32}$$

where $\epsilon_2 > 0$ is sufficiently small constant. By using Lemma 2.3 to (2.2.32) gives

$$\| \xi(\mu) - y(\mu) \| \le \gamma^{-1}\left[\gamma(\epsilon_2) + \int_0^\mu \frac{M\left\{\overline{p^{(n)}}(s)(1+\overline{q}(s)) + \overline{p^{(n)}}(s+1)\right\}}{1-M\overline{G}} ds\frac{1}{\zeta^*}\right] \tag{2.2.33}$$

Using the estimations in (2.2.29), (2.2.33) to the uniqueness problem, we use the notation $\gamma(r, r_0)$ instead of $\gamma(r)$ and impose the assumption $\lim_{r_0\to+0}\gamma(r, r_0) = +\infty$, for fixed r, then we obtain $\lim_{r_0\to+0}\gamma^{-1}(r, r_0) = 0$. From (2.2.29), (2.2.33), it follows that $\| \xi(\mu) - y(\mu) \| \le 0$ for $\mu \in R_+$ and hence $\xi(\mu) = y(\mu)$ on R_+. Thus, there is at most one solution to (1.1) (1.3) on R_+.

3 Application

Now we will discuss the application of some of our main theorems. we consider the fractional Volterra partial integrodifferential equation

$$\frac{\partial^\alpha \varphi(\mu,u)}{\partial \mu^\alpha} = \frac{\partial^2 \varphi(\mu,u)}{\partial u^2} + \eta(\mu,\varphi(\mu,u),\int_0^\mu k_1(\mu,s,\varphi(s,u))ds,\Theta(\mu)), \quad (3.1)$$

$$0 \le u \le 2\pi, \quad \mu \in [0,b],(b>0)$$

$$\varphi(\mu,0) = \varphi(\mu,2\pi) = 0, \mu \in [0,b],(b>0) \quad (3.2)$$

$$\varphi(0,u) + \sum_{i=1}^{q} \nu_i \varphi(\mu_i,u) = \xi_0(u),\ u \in [0,2\pi], \quad (3.3)$$

where $\eta : [0,b] \times R \times R \to R, k_1 : [0,b] \times [0,b] \times R \to R$ are continuous and $0 < \mu_i, s_i < b, \nu_i, \in R$ are prefixed numbers.
Let $\mathcal{X} = L^2([0,2\pi])$ and define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by $\mathcal{A}\varphi = \varphi_{uu}$, $D(\mathcal{A}) = \{\varphi(.) \in \mathcal{X} : \varphi(0) - \varphi(2\pi) = 0\}$. It is clearly $\mathcal{A}$ is the generator of $\{C(\mu) : \mu \in R\}$ on $\mathcal{X}$.
The functions, $\psi : [0,b] \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}, k_1 : [0,b] \times [0,b] \times \mathcal{X} \to \mathcal{X}$, and $p : ([0,b],\mathcal{X}) \to \mathcal{X}$ as follows

$$\psi(\mu,\xi,y)u = \eta(\mu,\xi(u),y(u)),$$

$$k(\mu,s,\xi)u = k_1(\mu,s,\xi(u)),$$

$$p(x)u = \sum_{i=1}^{n} \nu_i \xi(t_i,u),\ u \in C([0,2\pi] : X),$$

for $0 <, \mu_i < s_i < b$ and $0 \le u \le 2\pi$. We suppose that the functions k_1 in (3.1) satisfy all the hypotheses of the Theorem 2.4. Also we assume that

$$\mid \sum_{i=1}^{l} \varphi(\mu_i,u) - \sum_{i=1}^{l} \varphi(\mu_i,u) \mid \le G^* \sup_{\mu \in [0,b]} \mid u(\mu) - v(\mu) \mid$$

for all $u, v \in E_1 = C([0,b];R)$ and some constant $G^* > 0$. Then the above problem (3.1) -(3.3) can be formulated abstractly a quasilinear integrodifferential equation in Banach space X:

$$\xi^\alpha(\mu) + \mathcal{A}\xi(\mu) = \psi(\mu,\xi(\mu),\int_0^\mu K(\mu,s,\xi(s))ds,\xi(\mu-1))$$

for $\mu \in J = [0,b],(b>0)$ under the conditions

$$\xi(\mu-1) = \phi(\mu) \quad (0 \le \mu < 1),$$

$$\xi(0) + g(\xi) = \xi_0$$

Since all the hypotheses of the Theorem 2.4 are satisfied, the Theorem 2.4 can be applied to the mild solution of the fractional Volterra partial integrodifferential equations (3.1) -(3.3)

4 Conclusions

Qualitative analysis of solutions for non-linear fractional integrodifferential equation with non-local condition have been studied by using Leray-Schauder alternative and the Bihari's integral inequality. Also we obtained the solutions of the integrodifferential equations in Caputo sense. Moreover, example demonstrated the validity of the obtained main result.

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