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The Realization of Hofstadter Butterfly spectrum in the Microwave

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Abstract

The Hofstadter butterfly is the emblem of a long-standing problem concerning the single-electron states in a periodic electric potential and a transverse, constant magnetic field.The experimental electromagnetic realization of the Hofstadter butterfly was done by studying the transmission of microwaves through an array of 100 scatterers inserted into a waveguide when the modulation length of this uni-dimensional periodic lattice changed.We demonstrate the method of calculating similar spectra for finite lattices in a magnetic field, by using the methods of transmission microwaves through an array of scatterersinterpolate into a waveguide. For periodic sequences with varying period length the transmission bands reproduced the Hofstadter butterfly, originally predicted for the spectra of conduction electrons in strong magnetic fields. In the experiment it was used that the same transfer matrix formalism is applicable to both the microwave and the electronic systems.

Keywords:Hofstadter butterfly, microwave, transfer matrix

Acknowledgement

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I am truly grateful for all the support and inspirations that all of you had given me and will my best in everything I do in the future.

In 1955, P.G. Harper wrote a paper in which he develops the effects of the lattice potential and an external magnetic field on a two dimensional square lattice, is known as the ***Harper's equation***. However, Harper did not succeed in finding the solution of his equation for different values of the magnetic field. In 1976, Douglas Hofstadter [1]published a paper in which he performs the calculation of Harper's equation. The combined effect of the potential and the magnetic field gives rise to a quantization of the energy that exhibits a fractal pattern called Hofstadter butterfly, which describes the spectral properties of non-interacting two-dimensional electrons in a magnetic field in a lattice.

In the tight binding approximation our Schrödinger Equation turns into a one-dimensional difference equation:

$$\psi_{m+1} + \psi_{m-1} + 2\cos(2\pi m\alpha - \nu)\,\psi_m = \epsilon\psi_m \quad (1)$$

This equation is called "Harper's" equation, and has been studied by a number of authors. whereψ_m is the wave function at site m, and ν is a phase associated with the linear momentum of the electron, superimposed on the cyclotron orbits in the magnetic field. $\epsilon$is the energy in normalized units.

The spectra contain one single parameter$\alpha = eBd^2/hc$ counting the number of magnetic flux quanta per unit cell.All possible eigenenergiesof the Harper equation are in the range of ϵ between $-4$and $+4$, and value of α is in any unit interval. We shall look at the interval [0,1] which is sufficient to consider the range
$0 \leq \alpha \leq 1/2$,as the Harper equation is invariant under the substitution $\alpha \rightarrow 1-\alpha$.
Another way of writing Eq. (1) is:

$$\begin{pmatrix} \psi_{m+1} \\ \psi_m \end{pmatrix} = T_n \begin{pmatrix} \psi_m \\ \psi_{m-1} \end{pmatrix} \qquad (2)$$

where$T_n$is the transfer matrix given by

$$T_n = \begin{pmatrix} \epsilon - 2\cos(2\pi m\alpha - \nu) & -1 \\ 1 & 0 \end{pmatrix} \qquad (3)$$

The set of all eigenvalues of the Harper equation if plottedin the (ϵ, α)plane form the Hofstadter butterfly.

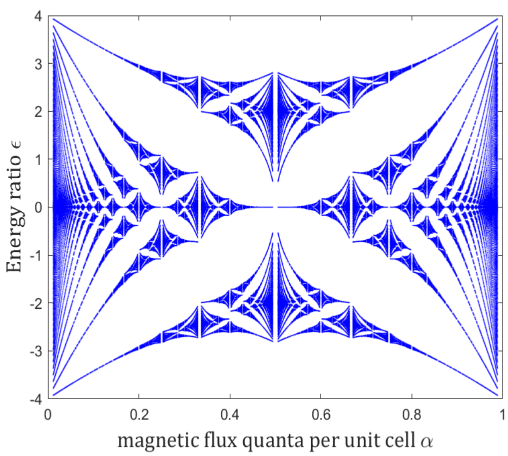

Fig.1.The Hofstadter butterfly, showing the eigenvalue spectrum of an electron on a square lattice with a perpendicularly applied magnetic field. The energies are plotted on the ordinate and range from-4 to $+4$. On the abscissa the number of flux quanta per unit cell α is shown between 0 and 1. For rational numbers $\alpha = eBd^2/hc = p/q$. The spectrum is made up of q bands, for irrational α the eigenvalues form a Cantor set.

For every rational value $\alpha = p/q$, the magnetic unit cell is by a factor q larger than the lattice unit cell causing a splitting of all electronics Bloch bands into q sub-bands,In other words,a single band for Zero magnetic fields splits into q bands when magnetic flux is a rational number With denominator q. For irrational value of α the spectra become fractal, and we have infinity of bands known as a Cantor set. It turns out that for each irrational the sum of the band widths is zero that is; the Cantor set has a zero measure. The resulting in unusual levelspacing distributions and diffusion properties [2] (for areview see Ref. [3]).For an experimental realization of the butterfly with typical lattice spacing of some 0.1nm magnetic fieldsof about 10^5T are necessary, which is far beyond the technically accessible limit.

The only possible technique to find a way around this problem is use artificial super-lattices, matching the structural periodicity with physical length scale of superconductivity and magnetism. In fact the first indications of a magnetic induced sub-bandsplitting due to commensurability effects have beenfound [4,5].Another possible system is Wigner crystals formedby the crystallization of a two-dimensional electrongas in a strong magnetic field due to Coulomb interaction. As the lattice constant of the Wigner crystal depends on the electron filling factor of the band,here too, commensurability effects are expected and, indeed observed [6]. In the mentioned works the splittingof an electron band into two sub-bands is reported, but measurements over a larger α range were not performed which is indispensable for an unequivocal identificationof the butterfly.

In the present work,we have proposed completely different idea is used, which capitulate the butterfly over the complete α and eigenenergy range.The procedures are based on the coincidencebetween electronic and photonic systems where the hypothesis is developed in solid state physics, such as band structures and localization are involved to the propagation of the electromagnetic waves through periodic and random systems [7].The microwave regime offers the most convenient experimental sample size to build and test formation of a complete photonic band gaps in dielectric structures have already been carry out by E.Yablonovitch and F.Gmitter[8] as well as by McCall *et al.*[9], J.Krug [10] and later R.E.Prange and Shmuel Fishman [11] prefer to study kicked quantum systems over the propagation of light through optical fibres with modulated refraction indices.

In the present Letter, a contiguous equivalence of the Harper equation,and an equation describing the propagation of waves (in our case microwaves) in a one-dimensional array of scatterers is speculated.Essentially the same transfer matrix technique as for the Harper equation can also be used to describe the propagation of microwaves through a scattering array. In a one-dimensional wave-guide with only one possible mode the amplitudes a_n, b_n of the waves propagating to the left and to the right respectively, are obtained from

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = T_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} \qquad (4)$$

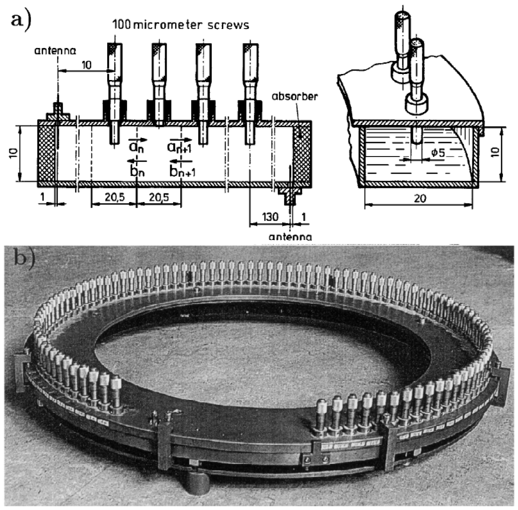

Fig. 2. (a) Schematic view of the waveguide *(a=20 mm,b=10 mm).* The microwaves are coupled in through theleft antenna, and the transmission through the waveguide with100 scatterers(micrometre screws) is measured with the rightantenna. At each end the waveguide is closed by microwaveabsorbers. (b) Photograph of the apparatus.

whereT_n is the transfer matrix associated with then^{th}scatterer [see in Fig.1(a)].From time-reversal symmetry displaces that the transfer matrix is given by:

$$T_n = \begin{pmatrix} \frac{1}{|t_n|} e^{i(\theta+\gamma_n)} & i\frac{|r_n|}{|t_n|} e^{-i\theta} \\ -i\frac{|r_n|}{|t_n|} e^{i\theta} & \frac{1}{|t_n|} e^{-i(\theta+\gamma_n)} \end{pmatrix} \quad (5)$$

where$|t_n|$ and $|r_n|$are the moduli of transmissionand reflection probabilities that an electron conservation requires that the probability of transmission plus the probability of reflection be unity, i.e 1=$|t_n|^2$+$|r_n|^2$,The probability for reflection *R* and transmission *T* are given by the usual quantum mechanics rule:

R=$|r_n|^2$ and T=$|t_n|^2$

In general, both *R* and *T* will be functions of the wavenumber*k*. $\gamma_n$is the phase of the transmission amplitude.

$\theta = kd/2\pi$is the phase shift from the free propagation between the scatterers.where*d* is the distance between the scatterers, and *k* the wave number.

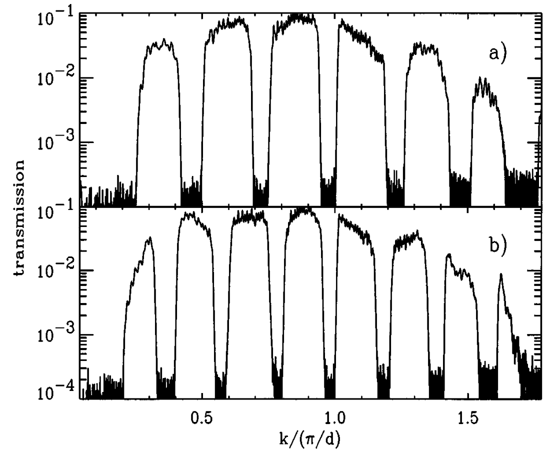

Fig.3. Transmission probability for a periodic scatterer arrangement showing the forbidden and allowed Bloch bands. In (a) every third and in (b) every fourth scatterer was introduced 3 mm. The shown wave number range corresponds to the frequency range from 7.5 to 15 GHz

Two different situations have to be discriminated,For the case $|Tr(T_n)|$< 2 both transfer matrices (3) and (5) can be written as:

$$T_n = e^{i\phi_n\sigma_n} \quad (6)$$

where

$$\cos\phi_n = \frac{\cos(\theta+\gamma_n)}{|t_n|}, \quad (7)$$

$$\sigma_n = \frac{1}{\sin\phi_n}\begin{pmatrix} \frac{\sin(\theta+\gamma_n)}{|t_n|} & \frac{|r_n|}{|t_n|} e^{-i\theta} \\ -\frac{|r_n|}{|t_n|} e^{i\theta} & \frac{-\sin(\theta+\gamma_n)}{|t_n|} \end{pmatrix} (8)$$

The trace of the transfer matrix contains all information, for the

scattering system and for the Harper equation.

Then the transfermatrix$T_n$can again be written in terms of a spinor rotation operator.

$$T_n = e^{i\phi_n\sigma_n} \quad (9)$$

where now

$$\cos\phi_n = \frac{E}{2} - \cos(2\pi m\alpha - \nu), (10)$$

$$\sigma_n = \frac{1}{i\sin\phi_n}\begin{pmatrix} \cos\phi_n & -1 \\ 1 & \cos\phi_n \end{pmatrix} \quad (11)$$

sinceσobeys the relation $(\sigma_n)^2 = 1$ and the eigenvalue of σ_n is $+1$ and -1. Hence,it's possible to transform σ_n to the spin matrix σ_z. For the case $|Tr(T_n)|$< 2 the ψ_n along the chain of sites are consequently obtained by successive spinor rotations. If all $T_n$are equal, which is the case for $\alpha = 0$, all rotations are about the same axis, and only the phase of the wave function changes from site to site, this is the range of the allowed electronic bands.

For the case $|Tr(T_n)|>2$, on the other hand, ϕ_n becomes imaginary, and the wave functions are exponentially damped. The same argumentation can be applied to the case of rational α value *p/q*. Now the sequence $\{T_n\}$ of transfer matrices is periodic with a period length q. In this case the value of the trace of the product $\prod_{n=1}^{q} T_n$ discriminates between the ranges of allowed and forbidden bands. The equivalence between the Harper equations (1) and (2) and Eq. (4) describing the propagation of microwaves through a one dimensional array of scatterers that is the basis for the microwave realization of the Hofstadter butterfly. i.e. microwave analogue experiment present another possibility to study the Hofstadter butterfly. The forms of the$\sigma_n$areinconsistent in the two cases, but this changes only the quantitative behaviour.The qualitative aspect of the spectra, the number of sub-bands, fractality, etc., is independent of the perfect form ofσ_n.

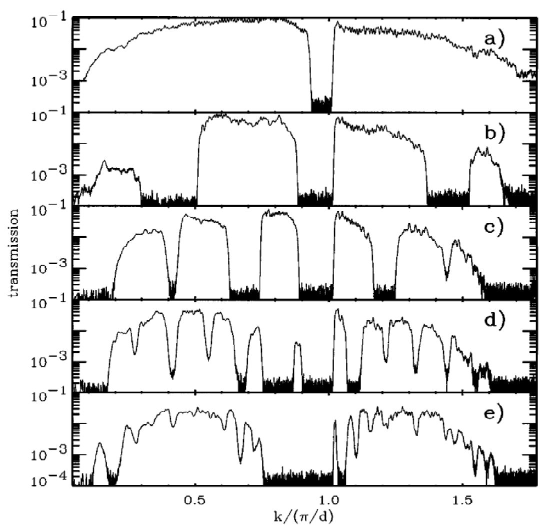

Fig.4. Transmission spectra for different periodic arrangements with $\alpha=1/q$,where$q$=1(a),2(b), 4(c), 8(d) and (e) 16. In (a) the two main Bloch bands can be seen each of which splits into q=2 sub-bands in (b) For larger values of q further band splitting is observed, but due to the strong absorptionin the system not all q sub-bands can be seen.

In the experiment a rectangular waveguide with dimensions *a=20mm,b=10mm*was used 100 cylindrical scatterers with a radius of *r*=2.5*mm*, and with the separation of d =20.5 mm, could be introduced into the waveguide (see in Fig.2).The lengths of the scatterers could be varied with the help of micrometre screws. The upper part of the waveguide could be rotated against the lower one, thus allowing one to vary the position of the exit antenna a feature which, however, was not yet used The total transmission through the empty waveguide (with all scatterers removed) amounted to only about 15%. This rather high loss is partly due to a mismatch of the antennas, and partly due to the absorbers at the end of the waveguide. If these effects are considered, one obtains for the real transmission through the waveguide a value of about 70%. In the figures below the uncorrected transmission probabilities are displayed. We measured in the frequency range where only the first mode can propagate, starting from the cut-off frequency of $\upsilon_{min}=\frac{c}{2a}\approx$7.5GHz up to $\upsilon_{max}=\frac{c}{a}=\frac{c}{2b}\approx$ 15GHz, where the propagation of the second mode becomes possible,

In a microwave wave-guide the dispersion relation is given by$k=\frac{2\pi}{c}\sqrt{\upsilon^2-\upsilon_{min}^2}$ .To escapea distortion of the spectra for low frequencies, for all data presented the wave number k is used as abscissa. in units of π/d, where d is the distance between the scatterers. To test the apparatus we started with periodic scattering arrangements. As an example Fig. 3.shows two transmission measurements demonstrating clearly the allowed and forbidden transmission bands. In the context of the above-mentioned analogy one may look upon the waveguide with a periodic array of scatterers as a one-dimensional photonic crystal [8].

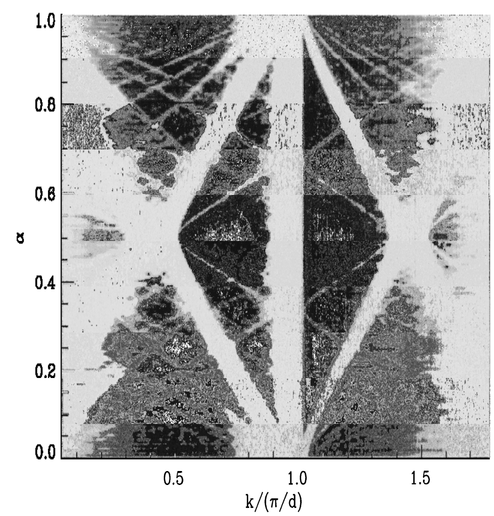

Fig. 5.Transmission spectra for a periodic arrangement of scatterers with α ranging from 0 to 1 in steps of 0.005. The upper part was obtained by reflection. All 100 scatterers were used. The first two Bloch bands are seen, showing two copies of the Hofstadter butterfly. The spectra were converted to a grey scale, where black and white corresponds to high and low transmission, respectively.

For the realization of the butterfly a periodic modulation of the lengths of the scatterers was applied with the period length as a parameter. We did not vary, however, the lengths according to
$l_n=l_0\cos(2\pi m\alpha-\nu)$ but replaced the cosinusoidal variation by a rectangular one.

$$l_n=\begin{cases}0 \; for \; \cos(2\pi m\alpha-\nu)\leq 0,\\ l_0 \; for \; \cos(2\pi m\alpha-\nu)>0\end{cases}$$

which was much easier to realize. In the experiment we choose $l_n = 3mm$ and $\nu = 0$ .Fig.4. shows a selection of transmission spectra for different $\alpha = 1/q$ values. In 4(a) the first two fundamental transmission bands are seen, corresponding to the first two Brillouin zones of the photonic crystal. In 4(b)-4(e) one finds the expected splitting into sub-bands, although because of the increasing absorption in the lower and upper parts of the frequency range not all expected q sub-bands are observed. The spectra are easier to interpret if the transmission probabilities are converted to a grey scale, and the spectra for different values of α are plotted together. This has been done in the Fig.5.

Now the structure of the Hofstadter butterfly is identified beyond all doubt, though the low and the high wave number regions are disturbed by absorption. Both two fundamental Bloch bands show a cross like sub-band splitting with α.thiscross is the dominant structure of the Hofstadter butterfly.In Fig. 6 spectra are shown where only every secondscatterer was used. Therefore now four Bloch bands are seen. Since only 50 scatterers were used, now less fractal details are seen as in Fig. 5, but instead the Hofstadter butterflies in the two middle Bloch bands are complete, and only the two outer bands are again disturbed by absorption. As the experiment allows an easy realization of arbitrary scattering arrangements, questions of transmission and localization in random or pseudorandom [12] structures may be studied equally well with the same apparatus.The possibility to also measure the field distribution along the waveguide is an additional interesting feature.Our doubts in the beginning that the absorption could prevent meaningful results fortunately showed up to be unfounded The experiment profited much from discussions withS. Fishman, Haifa, on possible realizations of kicked systems using microwave waveguide

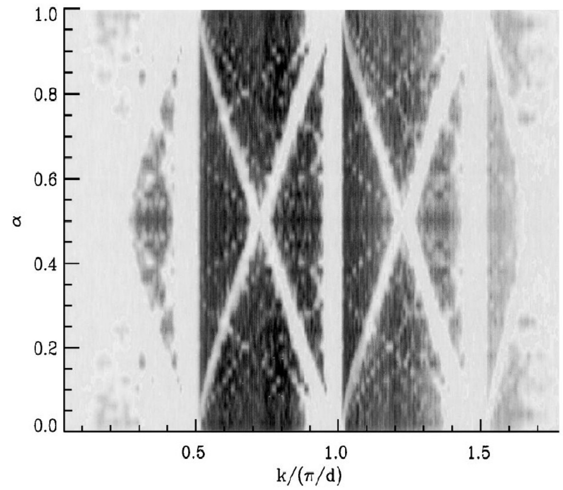

Fig. 6. As Fig. 4, but now α was varied in steps of 0.02, and only every second scatterer was used. Four Bloch bands are seen, each showing the Hofstadter butterfly.

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