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Velocity Distribution of Fluid Flow in Renal Tubules

Atinafu Asfaw Yesho
Department of Mathematics, Bonga University, Bonga, Ethiopia
Email: atinafu.asfaw2008@gmail.com

Abstract

In this paper we have develop mathematical modeling for fluid flow through renal tubules by approximating it to two dimensional flow. The model equation contains system of partial differential equation which is solved by method of perturbation method. Finally we have discussed the behavior of fluid flow and velocity distribution in renal tubules by approximating in two dimension and by taking different values for parametersRe, Reynolds number, and δ that is small parameter (the ratio of half width of the tubule to its length).

Key words: Flood flow, Perturbation method, Reynolds number

1.1. INTRODUCTION

The functional unit of the kidney is called nephron or renal tubule, each kidney has about 1 million of these tubules.One major part of a nephron is the glomerular tuft through which blood coming from the renal artery and afferent arterioles is filtered. Theglomerular filtrate is essential identical to plasma, and no chemical separation occurs up to this point. If the kidneys deliver this filtrate for excretion, the body losses many valuable materials,including water, at a rate faster than the one at which they can be supplied by synthesis or feeding. The rest of the nephron therefore recovers these valuable materials and returns them to the plasma. Thus about 80 percent of the filtrate is reabsorbed in the proximal tubule, and of the remaining, about 95 percent is further reabsorbed by the end of the collecting ducts.

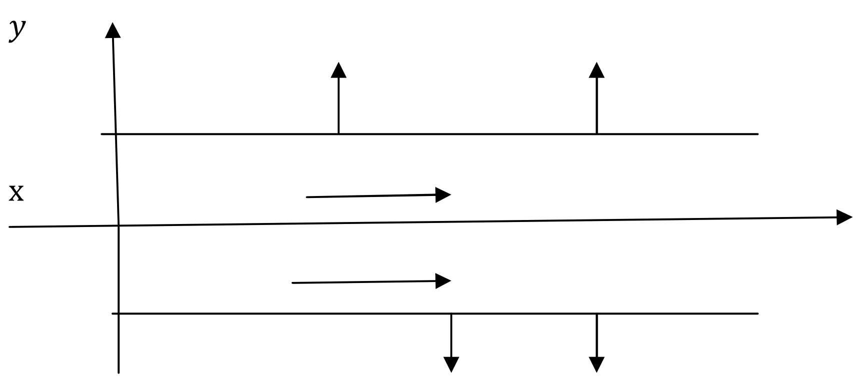

Fig1 approximation of fluid flow in renal tubules by two dimension

An extensive research work has been done on the fluid dynamics of biological fluid in presence and absence of magnetic field due to bio engineering and medical application. Masay (1969) study the mathematical modeling of the fluid flow in renal tubules and formulated the problem as the flow of an incompressible viscous fluid through a circular tube with linear rate of areabsorption at the wall. But in general this is not true, [Kelman (1974)], so in this seminar problemit is taken as exponentially decaying rate

In this seminar problem, we have made an attempt to understand the flow through renal tubule by studying the hydro dynamical aspects of an incompressible viscous fluid in uniform cross section with reabsorption at the walls. The boundary of channel walls assumed to be symmetric about x axis.

1.2. BASIC EQUATION AND BOUNDARY CONDITION

Consider incompressible fluid flow through channel of width $2a$and length $l$with uniform cross section. The motion of fluid assumed to be steady and symmetric about x axis. The governing equations of such fluid motion are given as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \ldots\ldots (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots\ldots (2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad \ldots\ldots (3)$$

Where u and v are the velocity components along x and y axis respectively, ρ density of fluid and $\nu=\frac{\mu}{\rho}$ is kinematic viscosity.

Boundary conditions are taken as follow:

a) No-slip condition:
   $u=0$ at$y=a$ ……. (4)
   Where$a$ is half width of channel
b) The regularity condition requires:

$v = 0$ and $\frac{\partial u}{\partial y} = 0$ at $y = 0$ ………………………………………………………….. (5)

c) The total flux across any section, i.e. the total volume of the fluid crossing any section per unit time is given as:

$Q(x) = \int_0^a u(x, y)dy = Q_0 F(\alpha x)$ ……………………………………………………….(6)

Where $F(\alpha x) = 1$ when $\alpha = 0$ and decrease with x, $\alpha \geq 0$ is the reabsorption coefficient and is constant, and Q_0 is the flux across the cross section at $x = 0$.

Let us introduce stream function ψ so that:

$u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ ………………………………………………………………….. (7)

Now, substituting equation (7) in equation (2) gives:

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + -\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right)$$

$$\Rightarrow \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right) \quad \ldots\ldots (8)$$

Again substituting equation (7) in equation (3) gives:

$$-\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu\left(\frac{\partial^3 \psi}{\partial y^2 \partial x} + \frac{\partial^3 \psi}{\partial x^3}\right) \quad \ldots\ldots (9)$$

To eliminate $p$from equation (8) and equation (9) we have to differentiate these equations with respect to y and x respectively.

So differentiating equation (8) with respect to y gives:

$$\frac{\partial^2 \psi}{\partial y^2}\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^3 \psi}{\partial y^2 \partial x}\frac{\partial \psi}{\partial y} - \frac{\partial^2 \psi}{\partial y^2}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^3 \psi}{\partial y^3}\frac{\partial \psi}{\partial x} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} + \nu\left(\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right)$$

$$\Rightarrow \frac{\partial^3 \psi}{\partial y^2 \partial x}\frac{\partial \psi}{\partial y} - \frac{\partial^3 \psi}{\partial y^3}\frac{\partial \psi}{\partial x} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} + \nu\left(\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) \quad \ldots\ldots (10)$$

Again differentiating equation (9) with respect to x gives:

$$-\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial^3\psi}{\partial x^3}\frac{\partial\psi}{\partial y}+\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\,\partial y}+\frac{\partial^3\psi}{\partial y\,\partial x^2}\frac{\partial\psi}{\partial x}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y\,\partial x}-\nu\left(\frac{\partial^4\psi}{\partial x^2\,\partial y^2}+\frac{\partial^4\psi}{\partial x^4}\right)$$

$$\Rightarrow-\frac{\partial^3\psi}{\partial x^3}\frac{\partial\psi}{\partial y}+\frac{\partial^3\psi}{\partial y\,\partial x^2}\frac{\partial\psi}{\partial x}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y\,\partial x}-\nu\left(\frac{\partial^4\psi}{\partial x^2\,\partial y^2}+\frac{\partial^4\psi}{\partial x^4}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (11)$$

Multiplying both sides of equation (11) by negative sign gives:

$$\frac{\partial^3\psi}{\partial x^3}\frac{\partial\psi}{\partial y}-\frac{\partial^3\psi}{\partial y\,\partial x^2}\frac{\partial\psi}{\partial x}=\frac{1}{\rho}\frac{\partial^2 p}{\partial y\,\partial x}+\nu\left(\frac{\partial^4\psi}{\partial x^2\,\partial y^2}+\frac{\partial^4\psi}{\partial x^4}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (12)$$

Now, adding equation (10) and equation (12) gives:

$$\frac{\partial\psi}{\partial y}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}=\nu\left(\frac{\partial^4\psi}{\partial y^4}+2\frac{\partial^4\psi}{\partial x^2\,\partial y^2}+\frac{\partial^4\psi}{\partial x^4}\right)$$

$$\Rightarrow\frac{\partial\psi}{\partial y}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}=\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (13)$$

Introducing the non-dimensional quantities:

$x^{'}=\frac{x}{l},y^{'}=\frac{y}{a},\psi^{'}=\frac{\psi}{Q_0},p^{'}=\frac{a^2}{\mu Q_0}p$ and substituting these quantities in equation (13) after dropping primes gives:

$$\frac{Q_0}{a}\frac{\partial\psi}{\partial y}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{Q_0}{l}\frac{\partial\psi}{\partial x}-\frac{Q_0}{l}\frac{\partial\psi}{\partial x}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{Q_0}{l}\frac{\partial\psi}{\partial y}=\nu Q_0\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)^2\psi$$

$$\Rightarrow\frac{{Q_0}^2}{al}\frac{\partial\psi}{\partial y}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{{Q_0}^2}{al}\frac{\partial\psi}{\partial x}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}=\nu Q_0\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)^2\psi$$

$$\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)^2\psi=\frac{Q_0}{al\nu}\left[\frac{\partial\psi}{\partial y}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\left(\frac{1}{l^2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}\right]$$

Multiplying both sides of above equation by a^4 gives:

$$\left(\frac{a^2}{l^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi=\frac{Q_0 a}{l\nu}\left[\frac{\partial\psi}{\partial y}\left(\frac{a^2}{l^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\left(\frac{a^2}{l^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}\right]$$

Let $\delta=\frac{a}{l}$ and $Re=\frac{Q_0}{\nu}$ substituting these in above gives:

$$\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi=\delta Re\left[\frac{\partial\psi}{\partial y}\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial y}\right]\ \ldots\ldots\ldots\ldots\ldots\ldots\ (14)$$

Further the boundary condition becomes:

$\frac{\partial \psi}{\partial y} = 0$ at $y = 1$ ……………………………………………………………………... (15)

$\psi = 0$ and $\frac{\partial^2 \psi}{\partial y^2} = 0$ at $y = 0$ ………………………………………………………………(16)

And

$\psi = F(\alpha x)$ at $y = 1$ ……………………………………………………………………… (17)

The parameter Re is Reynolds number and δ is small parameter (the ratio of half width of the tubule to its length). In this problem, we consider exponentially decaying bulk flow, that is,$F(\alpha x) = e^{-\alpha x}$ ……………………………………………………………………... (18)

1.3 Method of solution

It may be noted that the flow is quite complex because of non linearity of govern equation and boundary condition. Thus to solve equation we use perturbation theory which comprises mathematical method that are used to find analytic approximate solution to the problem which cannot be solved exactly, by starting from the exact solution of related problems.

Perturbation theory leads to an expression for the desired solution in terms of a formal power series in some "small" parameter known as a perturbation series. An approximate perturbation solution is obtained by truncating the series, usually by keeping the first two terms.

Now, for equation (14) we shall seek a solution for stream function $\psi(x, y)$ in the form of power series in terms of parameter δ and solving we get the solution as:

$$\Rightarrow p(x,y) = -Re\left(\int u\frac{\partial u}{\partial x}dx + \int v\frac{\partial u}{\partial y}dx\right) + \delta\frac{\partial u}{\partial x} + \frac{1}{\delta}\int\frac{\partial^2 u}{\partial y^2}dx \quad \text{…………………………} (19)$$

1.4 Results and discussion

The main objective of the study is to know the behavior of an incompressible fluid flow through the channel of uniform wall and the velocity distribution over such channel. We discuss the distribution of two velocity components $u(x, y)$ and $v(x, y)$ by fixing the parameters $\delta = 0.1, \alpha = 1.0\ and\ Re = 1.0$ in all our calculations.

Thus the values are obtained by taking different cross-sections of the channel$x = 0.2\ and\ x = 0.75$. By using C^{++} code for calculation we get the following values in table.

x	y	$u(x, y)$	$v(x, y)$
	0.0	0.909533	0.0
	0.2	0.873189	0.179417
0.5	0.4	0.764151	0.344321
	0.6	0.582389	0.480193
	0.8	0.327779	0.57248
	1.0	0.0	0.606531
	0.0	0.70839	0.0
	0.2	0.680077	0.139758
0.75	0.4	0.595135	0.268203
	0.6	0.453545	0.374018
	0.8	0.255231	0.445869
	1.0	0.0	0.472367

The above results in the table are expressed graphically as follows:

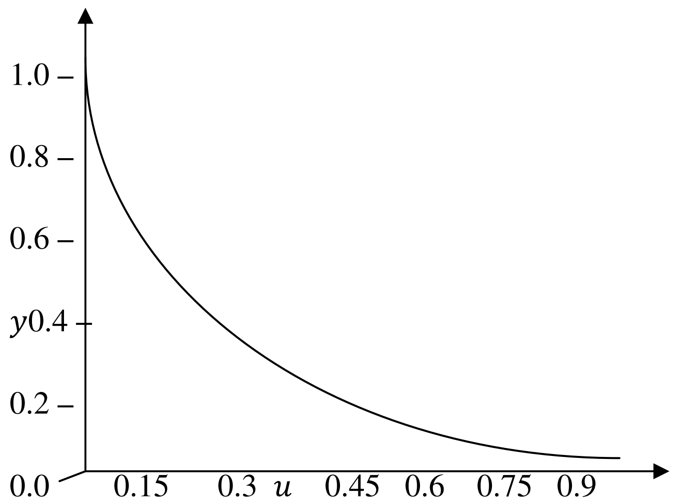

Figure 2(a). Distribution of velocity (u)with y ($x = 0.5$)

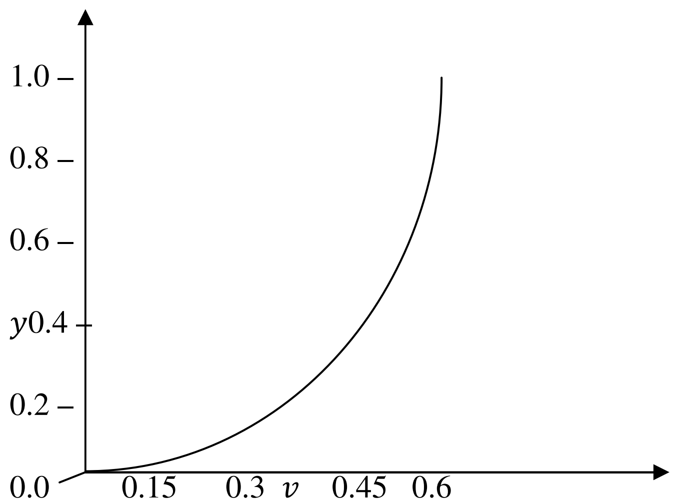

Figure 2(b). Distribution of velocity (v) with y ($x = 0.5$)

It can be observed that from figure 2(a)-2(b) that; the velocity u decreases with y while velocity v decreases. This indicates that fluid flow near wall of channel has less velocity along x direction than y direction and it becomes zero on the wall.

References

[1] Joel H: Ferziger.Milovan peric, "computational method for fluid dynamics" second edition (1999)

[2] Ronald L.Panton, "incompressible flow" third edition (1933)

[3] Stanley M.Walas, "modeling with differential equation in chemical engineering" (1991)

[5] JN Kapur, "mathematical modeling in biology and medicine"

