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A New Map on GTSs

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Abstract:

In this paper, we introduce one new function by using G_δ-sets and study their properties in a generalized topological space. Also, we prove every (μ, η)-continuous map is a (μ, η)-G_δ^* -map if (Y, η) is a generalized submaximal space. Finally, we give the relations between some functions with respect to different types of GT.

Keywords — open, dense, hyperconneted space, (μ, η)-continuous function.

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I. INTRODUCTION

The notion of a generalized topological space was introduced by Császár in [3]. Let X be any non-null set. A family $\mu \subseteq \exp(X)$ is a *generalized topology* [6] in X if $\varnothing \in \mu$ and $\bigcup_{t\in T} G_t \in \mu$ whenever $\{G_t \mid t \in T\} \subseteq \mu$ where $\exp(X)$ is a power set of X. We call the pair (X, μ) as a *generalized topological space* (GTS) [6]. If $X \in \mu$, then the pair (X, μ) is called a *strong generalized topological space* (sGTS) [6]. Let Y be a subset of X. Then the *subspace generalized topology* [2] is defined by, $\mu_Y = \{Y \cap U \mid U \in \mu\}$ and the pair (Y, μ_Y) is called as the *subspace generalized topological space* [2].

Let (X, μ) be a GTS and A be a subset of X. The *interior of A* [6] denoted by i_A, is the union of all μ-open sets contained in A and the *closure of A* [6] denoted by c_A, is the intersection of all μ-closed sets containing A when no confusion can arise. The elements in μ are called the μ-open sets, the complement of a μ-open set is called the μ-closed set and the complement of μ is denoted by μ'. Denote $\{U \in \mu \mid U \neq \varnothing\}$ by $\tilde{\mu}$ [5] and denote $\{U \in \mu \mid x \in U\}$ by $\mu(x)$ [5].

Throughout this paper, ℝ denote the set of all real numbers. The notations X_3, X_4 and X_5 are means that the sets $\{a, b, c\}$, $\{a, b, c, d\}$ and $\{a, b, c, d, e\}$, respectively.

II. PRELIMINARIES

In this section, we remember some basic definitions and lemmas which will be useful in the development of the next sections.

A subset A of a GTS (X, μ) is said to be a *μ-nowhere dense* [5] (resp. *μ-dense* [6], *μ-codense* [6]) set if $ic(A) = \varnothing$ (resp. $cA = X$, $c(X - A) = X$). A is said to be a *μ-strongly nowhere dense* set [5] if for every $V \in \tilde{\mu}$ there is $U \in \tilde{\mu}$ such that $U \subseteq V$ and $U \cap A = \varnothing$. Then A is said to be a *μ-meager* (or *μ-first category*) (resp. *μ-s-meager* (or *μ-s-first category*)) set [5] if $A = \bigcup_{n\in\mathbb{N}} A_n$ where A_n is a μ-nowhere dense (resp. μ-strongly nowhere dense) set for all $n \in \mathbb{N}$, ℕ is the set of all Natural numbers.

In a GTS, every subset of a μ-strongly nowhere dense set is a μ-nowhere dense set and every subset of a μ-meager (resp. μ-s-meager) set is μ-meager (resp. μ-s-meager) [5].

Let (*X*, μ) be a GTS and *A* be a subset of *X*. Then *A* is said to be a *μ-second category* (μ-II category) (resp. *μ-s-second category* (μ-s-II category)) set [5] if *A* is not a μ-meager (resp. μ-s-meager) set. *A* is a *μ-residual* (resp. *μ-s-residual*) [5] set if *X* - *A* is a μ-meager (resp. μ-s-meager) set.

A GTS (*X*, μ) is said to be *μ-II category* (resp. *μ-s-II category*) if *X* is μ-II category (resp. μ-s-II category) as a subset. A space *X* is called a *Baire space* (BS for short) (resp. *weak Baire space* (for short, wBS)) [5] if each $V \in \tilde{\mu}$ is of μ-II category (resp. μ-s-II category) in *X*. A space (*X*, μ) is a *strong Baire space*, in short sBS [5] if $V_1 \cap V_2 \cap \dots V_n$ is of μ-II category set for all $V_1, V_2, \dots, V_n \in \mu$ such that $V_1 \cap V_2 \cap \dots V_n \neq \emptyset$.

Also, every sBS is a BS and every BS is a wBS [5]. Equivalently, (*X*, μ) is called a *Baire space* [6] if for any sequence $\{G_n\}_{n\in\mathbb{N}}$ consisting of μ-open and μ-dense subsets of *X*, $\bigcap_{n\in\mathbb{N}} G_n$ is a μ-dense set in *X*.

Define $\mu^\star = \{\bigcup_t (U_1^t \cap U_2^t \cap \dots \cap U_{n_t}^t) \mid U_1^t, U_2^t, \dots, U_{n_t}^t \in \mu\}$ and $\mu^{\star\star} = \{A \subset X \mid A$ is of μ-II category set$\} \cup \{\emptyset\}$ [5]. Then $\mu \subset \mu^\star$ and $\mu \subset \mu^{\star\star}$ if (*X*, μ) is a Baire space [5]. Also, $\mu^\star \subset \mu^{\star\star}$ if (*X*, μ) is a sBS [9].

In a GTS, define $\mu^{\mathcal{V}} = \{\emptyset\} \cup \{A \subset X \mid A$ is of μ-s-II category set$\}$ [12]. Then $\mu^{\star\star} \subset \mu^{\mathcal{V}}$, $\mu \subset \mu^{\mathcal{V}}$ if (*X*, μ) is a wBS and $\mu^\star \subset \mu^{\mathcal{V}}$ if (X, μ) is a sBS [12].

A space (*X*, μ) is called *hyperconnected* [4] if every non-null μ-open subset of *X* is μ-dense in *X*. Let (*X*, μ) be a GTS. *X* is said to be a *generalized submaximal space* [4] if every μ-dense subset of *X* is a μ-open set in *X*.

The following lemmas will be useful in the sequel.

Lemma 2.1. [6, Lemma 3.2] Let (*X*, μ) be a GTS and let *A*, $U \subseteq X$. If $U \in \tilde{\mu}$ and $U \cap A = \emptyset$, then $U \cap cA = \emptyset$.

Lemma 2.2 [6, Theorem 5.3] Let (*X*, μ) be a GTS. The following are equivalent.
(a) *X* is Baire.
(b) If $A \neq \emptyset$ is μ-residual in *X*, then *A* is μ-dense in *X*.
(c) If $B \neq X$ is μ-meager in *X*, then *B* is μ-codense in *X*.
(d) Every $U \in \tilde{\mu}$ is μ-second category in *X*.
(e) $iF = \emptyset$, for every *F* is a μ-meager set in *X*.
(f) For every μ-closed set F_n with $iF_n = \emptyset$, $i(\bigcup_{n\in\mathbb{N}} F_n) = \emptyset$.

Lemma 2.3 [9, Theorem 4.3] Let (*X*, μ) be a hyperconnected space. If *X* is of μ-II category, then (*X*, μ) is a BS.

III. NATURE OF $G_\delta^\star$-MAP

In this section, we introduce one new map namely, $G_\delta^\star$-map and analyze the nature of this map in a generalized topological space.

First, we recall some basic definitions defined in a generalized topological space for improvement of this section.

A subset *A* of a generalized topological space (*X*, μ) is said to be a μ- G_δ *-set* [1] if $A = \bigcap_{n\in\mathbb{N}} A_n$ where each A_n is a μ-open set. A subset *B* of a generalized topological space (*X*, μ) is said to be a *μ-*F_σ*-set* [1] if $B = \bigcup_{n\in\mathbb{N}} B_n$ where each B_n is a μ-closed set.

A GTS (*X*, μ) is said to be a *generalized* G_δ*-submaximal space* [1] if every μ-dense subset of *X* is a μ-G_δ-set in *X*.

Now we define a new map namely, $G_\delta^\star$-map in a GTS as in the following way;

Definition 3.1. Let (*X*, μ), (*Y*, η) be two generalized topological spaces and $f: X \to Y$ be a map. Then *f* is said to be a (μ, η)- $G_\delta^\star$-map if $f^{-1}(V)$ is a μ-G_δ-set whenever *V* is a η-dense set.

The existence of a $G_\delta^\star$-map in a generalized topological space as shown by the following Example 3.2.

Example 3.2. Consider the generalized topological spaces (X_4, μ) and (X_5, η) where μ = {∅, {*a, b*}, {*a, b, c*}, {*a, b, d*}, {*a, c, d*}, {*b, c, d*}, *X*}, η = {∅, {*a, b, c*}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}}. Define a map $f: (X_4, \mu) \to (X_5, \eta)$ by *f*(*a*) = *b*; *f*(*b*) = *c*; *f*(*c*) = *d*; *f*(*d*)= *a*. Here $f^{-1}(V)$ is a μ-G_δ-set in X_4 whenever *V* is a η-dense set in X_5. Thus, *f* is a (μ, η)-$G_\delta^\star$-map.

Theorem 3.3. Let (*X*, μ), (*Y*, η) be two generalized topological spaces and $f: X \to Y$ be a map. Then the following are equivalent.

(a) *f* is a (*μ, η*)- $G_\delta^\star$-map.

(b) If *U* is a *η*-codense set in *Y,* then $f^{-1}(U)$ is a μ-F_σ-set in *X*.

Proof. (a) ⇒ (b). Let *U* be a η-codense set in *Y*. Then *Y* - *U* is a η-dense set in *Y* and so $f^{-1}(Y-U)$ is a μ-G_δ-set in *X*, by (a). Since $f^{-1}(Y-U) = X - f^{-1}(U)$ we have $X - f^{-1}(U)$ is a μ-G_δ-set in *X* which implies that $f^{-1}(U)$ is a μ-F_σ-set in *X*.

(b) ⇒ (a). Let *B* be a η-dense set in *Y*. Then *Y* - *B* is a η-codense set in *Y* and so $f^{-1}(Y-B)$ is a μ-F_σ-set in *X*, by (b). This implies $X - f^{-1}(B)$ is a μ-F_σ-set in *X* which implies that $f^{-1}(B)$ is a μ-G_δ-set in *X*. Therefore, *f* is a (*μ, η*)- $G_\delta^\star$-map.

Corollary 3.4. Let (*X*, μ), (*Y*, η) be two generalized topological spaces and $f: X \to Y$ be a map. If *f* is a (*μ, η*)- $G_\delta^\star$-map, then the following hold.

(a) If *A* is a η-nowhere dense set in *Y*, then $f^{-1}(A)$ is a μ-F_σ-set in *X*.

(b) If *B* is a η-strongly nowhere dense set in *Y*, then $f^{-1}(B)$ is a μ-F_σ-set in *X*.

Proof. We will present the detailed proof only for (a). Let *A* be a η-nowhere dense set in *Y*. Then $i_\eta c_\eta A = \emptyset$ and so $c_\eta(Y-A) = Y$ which implies that *A* is a η-codense set in *Y*. By hypothesis and Theorem 3.3, $f^{-1}(A)$ a μ-F_σ-set in *X*.

Corollary 3.5. Let (*X*, μ) be a GTS, (*Y*, η) be a BS and $f: X \to Y$ be a map. If *f* is a (μ, η)-$G_\delta^\star$-map, then the following hold.

(a) If *A* is a η-residual set in *Y*, then $f^{-1}(A)$ is a μ-G_δ-set in *X*.

(b) If *B* is a η-meager set in *Y*, then $f^{-1}(B)$ is a μ-F_σ-set in *X*.

Theorem 3.6. Let (*X*, μ) be a GTS, (*Y*, η) be a hyperconnecded space and $f: X \to Y$ be a map. If *Y* is of η-II category and if *f* is a (μ, η)-$G_\delta^\star$-mapping, then for any η-G_δ-set $A \subseteq Y$, $f^{-1}(A)$ is a μ-G_δ-set.

Proof. Let *A* be a η-G_δ-subset of *Y*. Then $A = \bigcap_{n\in\mathbb{N}} A_n$ where each A_n is a η-open set in *Y* and so A_n is a η-dense set in *Y* for all $n \in \mathbb{N}$, by hypothesis. Since (*Y*, η) is a hyperconnecded, η-II category space we have (*Y*, η) is a BS, by Lemma 2.3. Therefore, *A* is a η-dense set in *Y*. Hence $f^{-1}(A)$ is a μ-G_δ-set in *X*, since *f* is a (μ, η)-$G_\delta^\star$-map.

Theorem 3.7. Let (*X*, μ) be a hyperconnecded space, (*Y*, η) be a GTS and $f: X \to Y$ be a map. If *X* is of μ-II category and if *f* is a (μ, η)- $G_\delta^\star$-map, then $f^{-1}(A)$ is a μ-dense set in *X* whenever *A* is a η-dense subset of Y.

Proof. By hypothesis and Lemma 2.3, (*X*, μ) is a BS. Let *A* be a η-dense subset of *Y*. Then $f^{-1}(A)$ is a μ-G_δ-set in *X*, by hypothesis and so $f^{-1}(A) = \bigcap_{n\in\mathbb{N}} A_n$ where each A_n is a non-null μ-open set in *X*. By hypothesis, A_n is a μ-dense set in *X* for all

$n \in \mathbb{N}$ which implies that $f^{-1}(A)$ is a μ-dense set in *X*, since (*X*, μ) is a BS.

Theorem 3.8. Let (*X*, μ) be a GTS, (*Y*, η) be a generalized G_δ-submaximal space and $f: X \to Y$ be a map. If $f^{-1}(A)$ is a μ-G_δ-set whenever A is a η-G_δ -subset of *Y*, then *f* is a (μ, η)- $G_\delta^\star$-map.

Theorem 3.9. Let (*X*, η), (*Y*, μ) be two GTSs and $f: (X, \eta) \to (Y, \mu)$ be a map. If $(Y, \mu^{\star\star})$ is a generalized G_δ-submaximal, Baire space and the complement of a $\mu^{\star\star}$-G_δ-set is not a $\mu^{\star\star}$-G_δ-set, then the following are equivalent.
(a) *f* is a (η, $\mu^{\star\star}$)-$G_\delta^\star$-map.
(b) For any $\mu^{\star\star}$-G_δ- set A in *Y*, $f^{-1}(A)$ is a η-G_δ-set.

Proof. Suppose f is a (η, $\mu^{\star\star}$)-$G_\delta^\star$-map. Let A be a $\mu^{\star\star}$-G_δ-subset of *Y*. Then $A = \bigcap_{n\in\mathbb{N}} A_n$ where each A_n is a $\mu^{\star\star}$-open set and so Y - $A = \bigcup_{n\in\mathbb{N}}(Y - A_n)$ where each Y - A_n is a $\mu^{\star\star}$-closed subset of *Y*. Now we prove A is a $\mu^{\star\star}$-residual set in *Y*. Suppose $i_{\mu^{\star\star}}(Y - A_n) \neq \emptyset$ for some n. Then there exists $G \in \tilde{\mu}^{\star\star}$ such that $G \subseteq Y$ - A_n. Thus, G is of μ-II category so that Y - A_n is of μ-II category, since subset a meager set is a meager set. Therefore, Y - $A_n \in \tilde{\mu}^{\star\star}$ so that $Y - A \in \tilde{\mu}^{\star\star}$ Thus, Y - A is a $\mu^{\star\star}$-G_δ-set which is a contradiction to the hypothesis so that $i_{\mu^{\star\star}}(Y - A_n) = \emptyset$ for all n. Therefore, each Y - A_n is a $\mu^{\star\star}$-nowhere dense set. Hence Y - A is a $\mu^{\star\star}$-meager set implies that A is a $\mu^{\star\star}$-residual subset of *Y*. By hypothesis and Lemma 2.2, A is a $\mu^{\star\star}$-dense subset of *Y*. By assumption, $f^{-1}(A)$ is a η-G_δ-set. Converse follows from Theorem 3.8.

The condition ``the complement of a $\mu^{\star\star}$-G_δ-set is not a $\mu^{\star\star}$-G_δ-set" cannot be dropped in Theorem 3.9 as shown by the following Example 3.10.

Example 3.10. Consider the generalized topological spaces (X_3, η) and (X_4, μ) where η = {∅, {*a*, *b*}, {*a*, *c*}, X_3}, μ = {∅, {*a*, *b*}, {*a*, *b*, *c*}, {*a*, *b*, *d*}, X_4}. *Then* $\mu^{\star\star}$ = {∅} ∪ {{*a*}, {*b*}, {*a*, *b*}, {*a*, *c*}, {*a*, *d*}, {*b*, *c*}, {*b*, *d*}, {*a*, *b*, *c*}, {*a*, *b*, *d*}, {*a*, *c*, *d*}, {*b*, *c*, *d*}, X_4} and so (X_4, $\mu^{\star\star}$) is a generalized G_δ-submaximal, BS. But, there exist $U \subseteq X_4$ which is a $\mu^{\star\star}$-G_δ-set where U = {*c*, *d*} and also X_4 - U is a $\mu^{\star\star}$-G_δ-set. Define a map $f: (X_3, \eta) \to (X_4, \mu)$ by *f*(*a*) = *b*; *f*(*b*) = *c*; *f*(*c*) = *d*. Here $f^{-1}(V)$ is a η-G_δ-set in X_3 whenever V is a $\mu^{\star\star}$-dense set in X_4. Thus, f is a (η, $\mu^{\star\star}$)-$G_\delta^\star$-map. Let B = {c} be a subset of X_4. Then B is a $\mu^{\star\star}$-G_δ-set. Here $f^{-1}(B)$ = {b}. But {b} is not a η-G_δ-set in X_3. Thus, there exist $B \subseteq X_4$ which is a $\mu^{\star\star}$-G_δ-set set but $f^{-1}(B)$ is not a η-G_δ-set in X_3.

Theorem 3.11. Let *X* be a non-null set and μ, η be two generalized topologies defined in *X* such that μ ⊆ η. If *f* is either a (μ, η)- $G_\delta^\star$-map or (η, μ)- $G_\delta^\star$-map, then *f* is a (η, η)- $G_\delta^\star$-map.

Proof. Suppose f is a (μ, η)- $G_\delta^\star$-map. Let A be a η-dense set in *X*. Then $f^{-1}(A)$ is a μ-G_δ-set and so $f^{-1}(A)$ is a η-G_δ-set in *X*, since μ ⊆ η. Thus, *f* is a (η, η)- $G_\delta^\star$-map.
Assume that, (η, μ)- $G_\delta^\star$-map. Let A be a η-dense set in *X*. Since μ ⊆ η, η-dense is μ-dense. Thus, A is a μ-dense set in *X*. By assumption, $f^{-1}(A)$ is a η-G_δ-set in *X*. Therefore, *f* is a (η, η)-$G_\delta^\star$-map.

Observation 3.12. Let (*X*, μ) be a BS. If *f* is either a (μ, $\mu^{\star\star}$)- $G_\delta^\star$-map or ($\mu^{\star\star}$, μ)- $G_\delta^\star$-map, then f is a ($\mu^{\star\star}$, $\mu^{\star\star}$)- $G_\delta^\star$-map.

Proof. This observation follows from the fact that $\mu \subseteq \mu^{\star\star}$if (*X*, μ) is a BS and the similar arguments in Theorem 3.11 so the proof is omitted.

The following Observation 3.13 follows from the fact that $\mu \subseteq \mu^{\mathcal{V}}$ if (*X*, μ) is a wBS.

Observation 3.13. Let (*X*, μ) be a wBS. If *f* is either a (μ, $\mu^{\mathcal{V}}$)- $G_\delta^\star$-map or ($\mu^{\mathcal{V}}$, μ)- $G_\delta^\star$-map, then f is a ($\mu^{\mathcal{V}}$, $\mu^{\mathcal{V}}$)- $G_\delta^\star$-map.

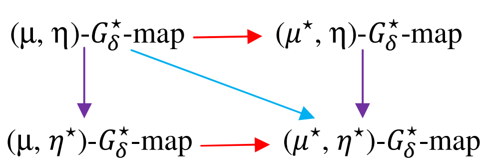
Observation 3.14. Let (*X*, μ) be a sBS. Then the following hold.

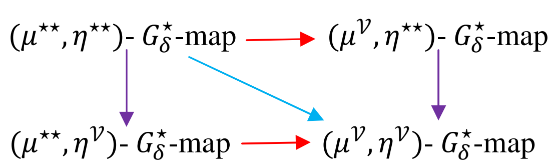
(a) If f is either a $(\mu^\star,\ \mu^{\star\star})$- $G_\delta^\star$-map or $(\mu^{\star\star},\ \mu^\star)$-$G_\delta^\star$-map, then f is a $(\mu^{\star\star},\ \mu^{\star\star})$- $G_\delta^\star$-map.
(b) If f is either a $(\mu^\star,\ \mu^{\mathcal{V}})$- $G_\delta^\star$-map or $(\mu^{\mathcal{V}},\ \mu^\star)$-$G_\delta^\star$-map, then f is a $(\mu^{\mathcal{V}},\ \mu^{\mathcal{V}})$- $G_\delta^\star$-map.

Proof. Suppose (*X*, μ) is a sBS. Then $\mu^\star \subseteq \mu^{\star\star}$ and $\mu^\star \subseteq \mu^{\mathcal{V}}$. By similar Considerations in Theorem 3.11, wet get the proof.

Remark 3.15. Let (*X*,μ) be a generalized topological space. Then the following hold.
(a) Every (μ, μ)-$G_\delta^\star$-map is a (μ, $\mu^\star$)- $G_\delta^\star$-map.
(b) In a BS, every (μ, μ)-$G_\delta^\star$-map is a (μ, $\mu^{\star\star}$)- $G_\delta^\star$-map.
(c) In a wBS, (μ, μ)-$G_\delta^\star$-map is a (μ, $\mu^{\mathcal{V}}$)- $G_\delta^\star$-map.
(d) Every $(\mu^{\star\star},\ \mu^{\star\star})$- $G_\delta^\star$-map is a $(\mu^{\star\star},\ \mu^{\mathcal{V}})$- $G_\delta^\star$-map.
(e) In a sBS, $(\mu^\star,\ \mu^\star)$- $G_\delta^\star$-map is a $(\mu^\star,\ \mu^{\mathcal{V}})$- $G_\delta^\star$-map and hence $(\mu^{\mathcal{V}},\ \mu^{\mathcal{V}})$- $G_\delta^\star$-map.

Let (*X*, μ) and (*Y*, η) be two generalized topological spaces. Then the following diagrams are always true.

1.


2.


Theorem 3.16. Let (*X*, μ) be a GTS, (*Y*, η) be a BS and $f: (X,\mu) \to (Y,\eta)$ be a map. If f is a (μ, η)-$G_\delta^\star$-map, then the following hold.
(a) f is a $(\mu,\eta^{\star\star})$- $G_\delta^\star$-map.
(b) f is a $(\mu,\ \eta^{\mathcal{V}})$- $G_\delta^\star$-map.
(c) f is a $(\mu^\star,\eta^{\star\star})$- $G_\delta^\star$-map.
(d) f is a $(\mu^\star,\eta^{\mathcal{V}})$- $G_\delta^\star$-map.

Proof. It is a direct consequence of the Definition 3.1 and using the facts that $\eta \subseteq \eta^{\star\star}$, $\eta \subseteq \eta^{\mathcal{V}}$.

In Theorem 3.16, we replace the condition ``(*Y*, η) be a BS" by ``(*Y*, η) be a wBS", then (b) and (d) are holds, only.

Theorem 3.17. Let (*X*, μ) be a GTS, (*Y*, η) be a sBS and $f: (X,\mu) \to (Y,\eta)$ be a map. If f is a $(\mu,\eta^\star)$-$G_\delta^\star$-map, then the following hold.
(a) f is a $(\mu,\ \eta^{\mathcal{V}})$- $G_\delta^\star$-map.
(b) f is a $(\mu,\eta^{\star\star})$- $G_\delta^\star$-map.
(c) f is a $(\mu^\star,\eta^{\mathcal{V}})$- $G_\delta^\star$-map.
(d) f is a $(\mu^\star,\eta^{\star\star})$- $G_\delta^\star$-map.

Theorem 3.18. Let (*X*, μ) be a sBS, (*Y*, η) be a GTS and $f: (X,\mu) \to (Y,\eta)$ be a map. If f is a $(\mu^\star,\eta)$-$G_\delta^\star$-map, then the following hold.
(a) f is a $(\mu^{\star\star},\eta)$- $G_\delta^\star$-map.
(b) f is a $(\mu^{\mathcal{V}},\eta)$- $G_\delta^\star$-map.
(c) f is a $(\mu^{\star\star},\ \eta^\star)$- $G_\delta^\star$-map.
(d) f is a $(\mu^{\mathcal{V}},\ \eta^\star)$- $G_\delta^\star$-map

A generalized topology μ on *X* is said to satisfy the *I-property* [8] whenever $W_1, W_2, \ldots, W_n \in \mu$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, $i_\mu(W_1 \cap W_2 \cap \ldots \cap W_n) \neq \emptyset$.

Theorem 3. 19. Let (X, μ) be a GTS and μ has the *I*-property. Then the following hold.
(a) Every μ-dense subset of *X* is a $\mu^\star$-dense set in *X*.
(b) Every μ-codense subset of *X* is a $\mu^\star$-codense set in *X*.
(c) Every μ-nowhere dense subset of *X* is a $\mu^\star$-codense set in *X*.
(d) Every μ-strongly nowhere dense subset of *X* is a $\mu^\star$-codense set in *X*.
(e) If (*X*, μ) is a hyperconnected space, then (*X*, $\mu^\star$) is a hyperconnected space.

Proof. It is enough to prove (a) and (e), only .
(a). Let *A* be a μ-dense subset of *X* and $G \in \tilde{\mu}^\star$. Then $G = \bigcup_t (G_1^t \cap G_2^t \cap \ldots \cap G_{n_t}^t)$ where $G_1^t, G_2^t, \ldots,$

$G^t_{n_t} \in \mu$. Take $G_k = G^k_1 \cap G^k_2 \cap \ldots \cap G^k_{n_k}$ where $G^k_1, G^k_2, \ldots, G^k_{n_k} \in \mu$ with $G_k \neq \emptyset$ for some k. By hypothesis, $i_\mu G_k \neq \emptyset$ which implies that $i_\mu G_k \in \tilde{\mu}$ which turn implies that $i_\mu G_k \cap A \neq \emptyset$. Thus, $G \cap A \neq \emptyset$. Therefore, A is a $\mu^\star$-dense set in X.

(e). Suppose (X, μ) is a hyperconnected space. Let $B \in \tilde{\mu}^\star$. Then $B = \cup_t(B^t_1 \cap B^t_2 \cap \ldots \cap B^t_{n_t})$ where $B^t_1, B^t_2, \ldots, B^t_{n_t} \in \mu$. Take $B_k = B^k_1 \cap B^k_2 \cap \ldots \cap B^k_{n_k}$ where $B^k_1, B^k_2, \ldots, B^k_{n_k} \in \mu$ with $B_k \neq \emptyset$ for some k. By hypothesis, $i_\mu B_k \neq \emptyset$ which implies that $i_\mu B_k \in \tilde{\mu}$ which turn implies that $i_\mu B_k$ is a μ-dense set. By (a), $i_\mu B_k$ is a $\mu^\star$-dense set in X implies that B_k is a $\mu^\star$-dense set in X. Therefore, B is a $\mu^\star$-dense set. Hence (X, $\mu^\star$)is a hyperconnected space.

Corollary 3.20. Let (X, μ), (Y, η) be a two GTSs and η has the I-property. If $f: (X, \mu) \rightarrow (Y, \eta)$ is a $(\mu, \eta^\star)$-$G^\star_\delta$-map, then the following hold.
(a) f is a (μ, η)-$G^\star_\delta$-map.
(b) f is a $(\mu^\star, \eta)$-$G^\star_\delta$-map.

Proof. This is direct consequence of Theorem 3.19 (a).

Corollary 3.21. Let (X, μ) be a BS, (Y, η) be a GTS and η has the I-property. If $f: (X, \mu) \rightarrow (Y, \eta)$ is a $(\mu, \eta^\star)$-$G^\star_\delta$-map, then the following hold.
(a) f is a $(\mu^{\star\star}, \eta)$-$G^\star_\delta$-map.
(b) f is a $(\mu^{\mathcal{V}}, \eta)$-$G^\star_\delta$-map.

Lemma 3.22. [10, Theorem 4.9.] Let (X, μ) be a generalized submaximal space. Then every μ-dense subset of X is a $\mu^{\star\star}$-dense set in X.

Theorem 3.23. Let (X, μ) be a generalized submaximal space. Then every μ-codense subset of X is a $\mu^{\star\star}$-codense set in X.

Proof. This proof directly follows from Lemma 3.22 so the proof is omitted.

The following Example 3.24 (a) shows that the condition ``(X, μ) be a generalized submaximal space" cannot be dropped in Lemma 3.22 and Theorem 3.23. The reverse implication of Lemma 3.22 and Theorem 3.23 is need not be true as shown by the following Example 3.24 (b).

Example 3.24. (a) Consider the generalized topological spaces (X_4, μ) where μ = {∅, {a, b}, {b, c}, {a, b, c}}. Then (X_4, μ) is not a generalized submaximal space. For, {b} is μ-dense but which is not μ-open. Here $\mu^{\star\star}$ = {∅} ∪ {A | b ∈ A}.

1. Take B = {a, c}. Then B is μ-dense but B is not $\mu^{\star\star}$-dense. For, $c_{\mu^{\star\star}}B$ = B ≠ X_4.

2. Let A = {b}. Then A is μ-codense but A is not $\mu^{\star\star}$-codense. For, $c_{\mu^{\star\star}}$(X_4 - A) = $c_{\mu^{\star\star}}$ ({a, c, d}) = {a, c, d} ≠ X_4.

(b) Consider the generalized topological spaces (X_4, μ) where μ = {∅, {a}, {a, b}, {b, c}, {a, c}, {a, b, c}, {a, c, d}, {a, b, d}, X_4}. Then (X_4, μ) is a generalized submaximal space. Then $\mu^{\star\star}$= {∅} ∪ {A | a ∈ A}.

1. Let G = {a}. Then G is $\mu^{\star\star}$-dense but G is not μ-dense. For, $c_\mu G$ = {a, d} ≠ X_4.

2. Let H = {b, c}. Then H is $\mu^{\star\star}$-codense but H is not μ-codense. For, c_μ(X_4 - H) = c_μ({a, d}) = {a, d} ≠ X_4.

Corollary 3.25. Let (X, μ) be a generalized submaximal space. Then the following hold.
(a) Every μ-nowhere dense subset of X is a $\mu^{\star\star}$-codense set in X.
(b) Every μ-strongly nowhere dense subset of X is a $\mu^{\star\star}$-codense set in X.

Proof. This is a direct consequence of Theorem 3.23 so the proof is neglated.

Let (*X*, μ) be a GTS and (*Y*, η) be a generalized submaximal space. Then by Lemma 3.22 the following diagram is true.

(μ, $\eta^{\star\star}$)- $G_\delta^\star$-map ⟶ (μ, η)- $G_\delta^\star$-map

(μ, $\eta^{\star\star}$)- $G_\delta^\star$-map ↘ ($\mu^\star, \eta$)- $G_\delta^\star$-map

Corollary 3.26. Let (*X*, μ) be a BS and (*Y*, η) be a generalized submaximal space. If $f: (X, \mu) \to (Y, \eta)$ is a (μ, $\eta^{\star\star}$)- $G_\delta^\star$-map, then the following hold.
(a)} *f* is a $(\mu^{\star\star}, \eta)$- $G_\delta^\star$-map.
(b) *f* is a $(\mu^{\mathcal{V}}, \eta)$- $G_\delta^\star$-map

Proof. This proof directly follows from Lemma 3.22 so the proof is omitted.

IV. FUNCTIONS ON GTSs

In this section, we give the relations between $G_\delta^\star$-map and continuous map, feebly open map in a GTS. Also, we analyze the nature of four types of functions in a generalized topological space.

First of all, we remember some basic definitions for the development of this section.

Let (*X*, μ) and (*Y*, η) be two GTSs. A function $f: (X, \mu) \to (Y, \eta)$ is called *feebly* (μ, η)-*continuous* [7] (resp. (μ, η)-*continuous* [6]) if $i_\mu\big(f^{-1}(B)\big) \neq \varnothing$ for every $B \subseteq Y$ with $i_\eta B \neq \varnothing$ (resp. $f^{-1}(U) \in \mu$ for each $U \in \eta$). Then *f* is called a (μ, η)-*open* (resp. *feebly* (μ, η)-*open*) map if $f(U) \in \eta$ for each $U \in \mu$ [6] (resp. $i_\eta\big(f(U)\big) \neq \varnothing$ for each $i_\mu U \neq \varnothing$ [7]).

Lemma 4.1 [10, Theorem 3.4] Let (*X*, μ) be a GTS. If *X* is of μ-II category, then the following hold.
(a) Every μ-meager subset of *X* is not a μ-residual set in *X*.
(b) Every μ-residual subset of *X* is a μ-II category set in *X*.
(c) Complement of a μ-nowhere dense set is a μ-II category set in *X*.

Lemma 4.2. [11, Theorem 4.6] Let (*X*, μ), (*Y*, η) be two GTSs and $f: (X, \mu) \to (Y, \eta)$ be a feebly (μ, η)-open map. If *A* is η-codense, then $f^{-1}(A)$ is μ-codense.

Theorem 4.3. Let (*X*, μ) be a generalized G_δ-submaximal space and (*Y*, η) be a GTS. If $f: (X, \mu) \to (Y, \eta)$ is a feebly (μ, η)-open map, then *f* is a (μ, η)-$G_\delta^\star$-map.

Proof. Let *A* be a η-dense set in *Y*. Then *Y* - *A* is a η-codense set in *Y* and so $f^{-1}(Y - A)$ is μ-codense in *X*, by Lemma 4.2. Thus, *X* - $f^{-1}(A)$ is a μ-codense set in *X* so that $f^{-1}(A)$ is μ-dense in *X*. By hypothesis, $f^{-1}(A)$ is a μ-G_δ-set in *X*. Hence *f* is a (μ, η)-$G_\delta^\star$-map.

In Example 4.4, (a) proves that the condition ``(*X*, μ) be a generalized G_δ-submaximal space" can not be dropped in Theorem 4.3 and (b) shows that the reverse implication of Theorem 4.3 is need not be true.

Example 4.4. (a). Consider the generalized topological spaces (X_4, μ) and (X_5, η) where μ = {∅, {*a*, *b*, *c*}, {*a*, *c*, *d*}, X_4}, η = {∅, {*a*, *b*}, {*b*, *c*}, {*a*, *b*, *c*}, {*a*, *b*, *d*}, {*a*, *b*, *c*, *d*}}. Then (X_4, μ) is not a generalized G_δ-submaximal space. For, let *A* = {*c*}. Then *A* is a μ-dense set. But *A* is not a μ-G_δ-set in X_4. Define a map $f: (X_4, \mu) \to (X_5, \eta)$ by *f*(*a*) = *b*; *f*(*b*) = *c*; *f*(*c*) = *a*; *f*(*d*) = *d*. Here $i_\eta\big(f(U)\big) \neq \varnothing$ for each $i_\mu U \neq \varnothing$. Thus, *f* is a feebly (μ, η)-open map. Let *U* = {*b*} be a subset of X_5. Then *U* is a η-dense set. But $f^{-1}(U)$ = {*a*} is not a μ-G_δ-set in X_4. Thus, *f* is not a (μ, η)-$G_\delta^\star$-map.

(b). Consider the generalized topological spaces (X_4, μ) and (X_5, η) where μ = {∅, {*a*, *b*}, {*b*, *c*}, {*a*, *b*, *c*}, {*a*, *b*, *d*}, {*a*, *c*, *d*}, {*b*, *c*, *d*}, X_4}, η = {∅, {*a*, *d*}, {*b*, *c*}, {*c*, *d*}, {*a*, *c*, *d*}, {*b*, *c*, *d*}, {*a*, *b*, *c*, *d*}}. Then (X_4, μ) is a generalized G_δ-submaxi-mal space. Define a map $f: (X_4, \mu) \to (X_5, \eta)$ by *f*(*a*) = *b*; *f*(*b*) = *a*; *f*(*c*) = *d*; *f*(*d*) = *c*. Here $f^{-1}(V)$ is a μ-G_δ-set in

X_4 whenever V is a η-dense set in X_5. Thus, f is a (μ, η)-$G_\delta^\star$-map. Choose $U = \{a, b\} \subseteq X_4$. Then $i_\mu U \neq \emptyset$. But $i_\eta f(U) = i_\eta(\{a, b\}) = \emptyset$. Thus, f is not a feebly (μ, η)-open mapping.

Theorem 4.5. Let (X, μ) be a GTS and (Y, η) be a generalized submaximal space. If $f: (X, \mu) \to (Y, \eta)$ is a (μ, η)-continuous map, then f is a (μ, η)-$G_\delta^\star$-map.

Proof. Let A be a η-dense set in Y. By hypothesis, A is η-open so that $f^{-1}(A)$ is a μ-open set in X, since f is a (μ, η)-continuous map. Thus, $f^{-1}(A)$ is a μ-G_δ-set in X. Hence f is a (μ, η)-$G_\delta^\star$-map.

Example 4.6 (a) proves that the condition ``(Y, η) be a generalized submaximal space" cannot be dropped in Theorem 4.5 and the converse part of Theorem 4.5 is need not be true as shown in Example 4.6 (b).

Example 4.6. (a). Consider the generalized topological spaces (X_3, μ) and (X_4, η) where μ = {∅, {a, c}, {b, c}, X_3}, η = {∅, {b, c}, {b, d}, {b, c, d}}. Then (X_4, η) is not a generalized submaximal space. For, let A = {a, b}. Then A is a η-dense set. But A is not a η-open set in X_4. Define a map $f: (X_3, \mu) \to (X_4, \eta)$ by $f(a) = c$; $f(b) = d$; $f(c) = b$. Here $f^{-1}(U) \in \mu$ for each $U \in \eta$. Thus, f is a (μ, η)-continuous map. Let U = {c, d} be a subset of X_4. Then U is η-dense. But $f^{-1}(U)$ = {a, b} is not a μ-G_δ-set in X_3. Thus, f is not a (μ, η)-$G_\delta^\star$-map.

(b) Consider the set X_4 with two generalized topologies μ, η where μ = {∅, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X_4}, η = {∅, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X_4}. Then (X_4, η) is a generalized submaximal space. Define a map $f: (X_4, \mu) \to (X_4, \eta)$ by $f(a) = d$; $f(b) = c$; $f(c) = b$; $f(d) = a$. Here $f^{-1}(V)$ is a μ-G_δ-set in X_4 whenever V is a η-dense set in X_4. Thus, f is a (μ, η)-$G_\delta^\star$-map. Choose U = {d} ⊆ X_4. Then $U \in$ η. But $f^{-1}(U) \notin \mu$. Thus, f is not a (μ, η)-continuous map.

Theorem 4.7. Inverse image of a generalized submaximal space is a generalized G_δ-submaximal space whenever the function is (μ, η)-continuous and an injective.

Proof. Let (X, μ) be a GTS and (Y, η) be a generalized submaximal space. Suppose $f: (X, \mu) \to (Y, \eta)$ is a (μ, η)-continuous, injective map. Then f is a (μ, η)-$G_\delta^\star$-map, by Theorem 4.5. Let A be a μ-dense subset of X. Then $f(A)$ is a η-dense subset of Y, since f is (μ, η)-continuous and so f^{-1}(f(A)) is a μ- G_δ -set in X, by hypothesis. Since f is an injective map, A is a μ-G_δ-set in X. Hence (X, μ) is a generalized G_δ-submaximal space.

Corollary 4.8. Let (X, μ) be a GTS and (Y, η) be a generalized submaximal space. If $f: (X, \mu) \to (Y, \eta)$ is a (μ, η)-continuous, injective map, then (X, $\mu^\star$) is a generalized G_δ-submaximal space.

Theorem 4.9. Inverse image of a hyperconnected space is hyperconnected under the feebly (μ, η)-open, injective map.

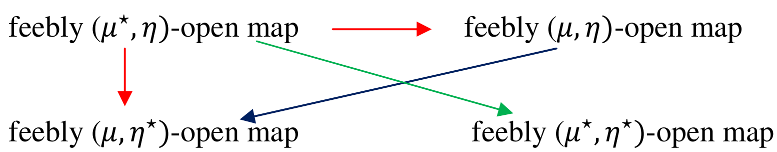
Proof. Let (X, μ) be a GTS and (Y, η) be a hyperconnected space. Suppose $f: (X, \mu) \to (Y, \eta)$ is a feebly (μ, η)-open, injective map. Let $G \in \tilde{\mu}$. By hypothesis, i_η (f(G)) ≠ ∅. Since (Y, η) is a hyperconnected space, i_η(f(G)) is a η-dense set and so f(G) is a η-dense set. Thus, Y - f(G) is a η-codense set so that f^{-1}(Y - f(G)) is a μ-codense set in X, by Lemma 4.2. Then X - f^{-1}(f(G)) is a μ-codense set and so f^{-1}(f(G)) is a μ-dense set in X. Since f is an injective map, G is a μ-dense set in X. Hence (X, μ) is a hyperconnected space.

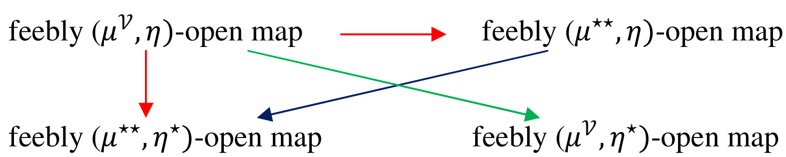
Theorem 4.10. Let (X, μ) be a GTS, (Y, η) be a hyperconnected space and $f: (X, \mu) \to (Y, \eta)$ be a feebly (μ, η)-open, injective map. If X is of μ-II category and if f is a (μ, η)-$G_\delta^\star$-map, then f is a

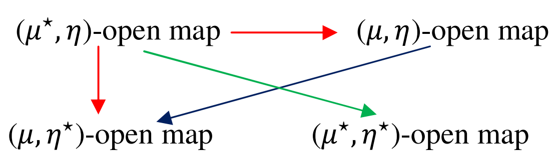
$(\mu^{\star\star},\eta)$-continuous map and hence $(\mu^{\mathcal{V}},\eta)$-continuous map.

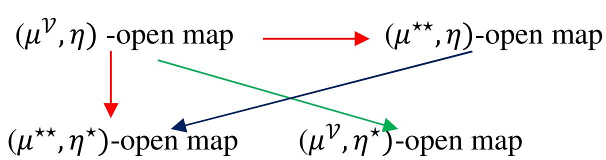
Proof. Let G be a η-open set in Y. Suppose $G = \varnothing$. Then $f^{-1}(G) = \varnothing$ and so $f^{-1}(G) \in \mu^{\star\star}$. Assume that, $G \neq \varnothing$. By hypothesis, G is a η-dense set in Y. Then $f^{-1}(G)$ is a μ-G_δ-set, since f is a (μ, η)-$G_\delta^\star$-map. Suppose $f^{-1}(G) = \varnothing$. Then there is nothing to prove. Assume that, $f^{-1}(G)$ is a non-null μ-G_δ-set. Then $f^{-1}(G) = \bigcap_{n\in\mathbb{N}} G_n$ where each $G_n \in \tilde{\mu}$. By hypothesis and Theorem 4.9, (X, μ) is a hyperconnected space so that each G_n is a μ-dense set. By hypothesis and Lemma 2.3, (X, μ) is a BS so that $f^{-1}(G)$ is a μ-dense set in X. Thus, $f^{-1}(G)$ is a μ-dense, μ-G_δ-set in X. Here each G_n is a μ-dense set so that $i_\mu(X - G_n) = \varnothing$. Also, $X - G_n$ is a μ-closed set in X for each $n \in \mathbb{N}$, since each $G_n \in \tilde{\mu}$. Therefore, each $X - G_n$ is a μ-nowhere dense set. Hence $X - f^{-1}(G)$ is a μ-meager set so that $f^{-1}(G)$ is a μ-residual set in X. By hypothesis and Lemma 4.1, $f^{-1}(G)$ is of μ-II category in X. Thus, $f^{-1}(G) \in \mu^{\star\star}$. Therefore, f is a $(\mu^{\star\star},\eta)$ -continuous map. Since $\mu^{\star\star} \subseteq \mu^{\mathcal{V}}$, f is a $(\mu^{\mathcal{V}},\eta)$-continuous map.

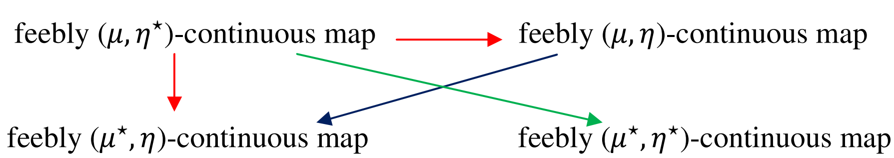
Let (X, μ) and (Y, η) be two GTSs. Then the following diagrams are always true.

1.


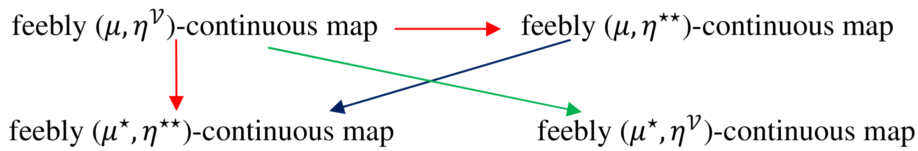
2.


3.


4.


5.


6.



In the rest of this section, we give some properties for cliquish functions in GTSs.

First of all, we remember the definition for cliquish function in a GTS.

Here, we will denote the set of μ-continuity (resp. μ-discontinuity) points of *f*, by $\mathcal{C}_{\mu}(f)$ (resp. $\mathcal{D}_{\mu}(f)$) where $f: X \to \mathbb{R}$.

A function $f : X \to \mathbb{R}$ is (μ, η)-*cliquish* if the set $\mathcal{C}_{\eta}(f)$ is μ-dense [5]. If (*X*, μ) is a BS and $\mathcal{D}_{\eta}(f)$ is a μ-meager set, then *f* is (μ, η)-cliquish [5].

Theorem 4.11. Let (*X*, μ) be a GTS and $f : X \to \mathbb{R}$ be a map. If μ has the *I*-property and if *f* is (μ, η)-cliquish, then *f* is a $(\mu^{\star}, \eta)$-cliquish function on *X* where η is any GT on *X*.

Proof. Suppose *f* is a (μ, η)-cliquish function on *X*. Then $\mathcal{C}_{\eta}(f)$ is μ-dense and so $\mathcal{C}_{\eta}(f)$ is $\mu^{\star}$-dense, by hypothesis and Theorem 3.19 (a). Hence *f* is a $(\mu^{\star}, \eta)$-cliquish function on *X*.

Theorem 4.12. Let (*X*, μ) be a generalized submaximal space and $f : X \to \mathbb{R}$ be a map. If *f* is (μ, η)-cliquish, then *f* is a $(\mu^{\star\star}, \eta)$-cliquish function on *X* where η is any GT on *X*.

Theorem 4.13. Let (*X*, μ) be a weak Baire space and $f : X \to \mathbb{R}$ be a map. If *f* is $(\mu^{\mathcal{V}}, \eta)$-cliquish, then *f* is a (μ, η)-cliquish function on *X* where η is any GT on *X*.

Theorem 4.14. Let (*X*, μ) be a generalized topological space and $f : X \to \mathbb{R}$ be a map. If $\mu^{\star\star} \neq \{\emptyset\}$ and if *f* is $(\mu^{\mathcal{V}}, \eta)$-cliquish, then *f* is a $(\mu^{\star\star}, \eta)$-cliquish function on *X* where η is any GT on *X*.

V. CONCLUSION

This paper provides the basic analysis for carrying out $G_{\delta}^{\star}$-map in a GTS. We have given a special kind of relationship between various types of functions in a GTS.

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