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Moment-Generating Functions and Reproductive Properties of Distributions

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Abstract:

In this paper, an important mathematical concept which has many applications to the probabilistic models are presented. Some of the important applications of the moment- generating function to the theory of probability are discussed. Each probability distribution has a unique moment-generating function, which means they are especially useful for solving problems like finding the distribution for sums of random variables. Reproductive properties of probability distributions with illustrated examples are also described.

Keywords **—Density function, distributions, moment-generating function, probability, random variable.**

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I. INTRODUCTION

In probability theory, an experiment with an outcome depending on chance which is called a *random experiment*. It is assumed that all possible distinct outcomes of a random experiment are known and that they are elements of a fundamental set known as the *sample space*. Each possible outcome is called a *sample point* and an *event* is generally referred to as a subset of the sample space having one or more sample points as its elements [5].

The behavior of a random variable is characterized by its probability distribution, that is, by the way probabilities are distributed over the values it assumes. A probability distribution function and a probability mass function are two ways to characterize this distribution for a discrete random variable. The corresponding functions for a continuous random variable are the probability distribution function (pdf) and the probability density function [5].

Assume that X is a random variable; that is, X is a function from the sample space to the real numbers. In computing various characteristics of the random variable X, such that $E(X)$ or $V(X)$, we work directly with the probability distribution of X. The moment-generating function $M_X(t)$ is the value which the function M_X is the value which the function M_X assumes for the real variable t. The notation, indicating the dependence on X, is used because we consider two random variables, X and Y, and then investigate the moment -generating function of each, M_X and M_Y. The moment-generating function is written as an infinite series or improper integral, depending on whether the random variable is discrete or continuous [4].

II. SOME DISTRIBUTION FUNCTIONS

In the nondeterministic or random mathematical models, parameters may also be used to characterize the probability distribution. With each probability distribution we may associate certain parameters which yields valuable information about the distribution [5].

A. Definition

Let X be a discrete random variable with possible value $x_1,\dots,x_n,\dots$. Let $p(x_i)=P(X=x_i)$, $i=1,2,\dots$. Then the *expected value* of X, denoted by $E(X)$, is defined as

$$E(X)=\sum_{i=1}^{\infty}x_i p(x_i)\ (1)$$

if the series $\sum_{i=1}^{\infty}x_i p(x_i)$ converges absolutely.

This number is also referred to as the *mean value* of X.

B. Definition

Let X be a continuous random variable with probability density function f. The *expected value* of X is defined as

$$E(X)=\int_{-\infty}^{+\infty}xf(x)dx, \qquad (2)$$

if the improper integral is a absolutely convergent, that is,

$$\int_{-\infty}^{+\infty}|x|f(x)dx.$$

C. Binomial Distribution

Consider an experiment ε and let A be some event associated with ε. Suppose that $P(A)=p$ and hence $P(\bar{A})=1-p$. Consider n independent repetitions of ε. Hence the sample space consists of all possible sequences $\{a_1,a_2,\dots,a_n\}$, where each a_i is either A or $\bar{A}$, depending on whether A or $\bar{A}$ occurred on the i^{th} repetition of ε. Furthermore, assume that $P(A)=p$ remains the same for all repetitions [2].

Let the random variable X be defined as follows: $X=$ number of times the event A occurred. Then X is called a *binomial* random variable with parameters n and p. Its possible values are obviously $0,1,2,\dots,n$. Equivalently X has a *binomial distribution*. The individual repetitions of ε will be called *Bernoulli trails*.

D.Uniform Distribution

Suppose that X is continuous random variable assuming all values in the interval $[a,b]$, where both a and b are finite. If the pdf of X is given by

$$F(x)=\frac{1}{b-a},\ a\le x\le b \qquad (3)$$
$$=0,\quad \text{elsewhere},$$

Then X is *uniformly distributed* over the interval $[a,b]$.

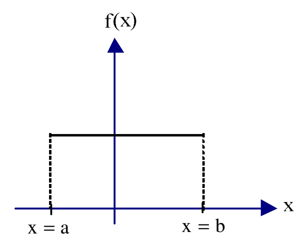

Fig. 1 X has uniformly distribution

E. The Poisson Distribution

Let X be a discrete random variable assuming the possible values: $0,1,2,\dots,n\dots$. If

$$P(X=k)=\frac{e^{-\alpha}\alpha^k}{k!},\ k=0,1,2,\dots,n,\dots. \qquad (4)$$

then X has a *Poisson distribution* with parameter $\alpha>0$.

F. Geometric Distribution

Assume, as in the discussion of the binomial distribution, that we perform ε repeatedly, that the repetitions are independent and that an each repetition $P(A)=p$ and $P(\bar{A})=1-p=q$ remain the same. Suppose that we repeat the experiment until

A occurs for the first time. Define the random variable X as the number the repetitions required up to and including the first occurrence of A. Thus X assumes the possible values $1,2,\ldots$. Since $X = k$ if and only if the first $(k-1)$ repetitions of ε result in $\bar{A}$ while the k^{th} repetition results in A,

$$P(X=k) = q^{k-1}p,\ k = 1,2,\ldots. \qquad (5)$$

A random variable with probability distribution (5) is said to have a *geometric distribution*[1].

G. The Normal Distribution

The random variable X, assuming all real values $-\infty < x < \infty$, has a *normal* (or *Gaussian*) *distribution* if its pdf is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right), -\infty < x < \infty. \qquad (6)$$

The parameters μ and σ must satisfy the conditions $-\infty < \mu < \infty, \sigma > 0$. X has distribution $N(\mu, \sigma^2)$ if and only if probability distribution is given by (6).

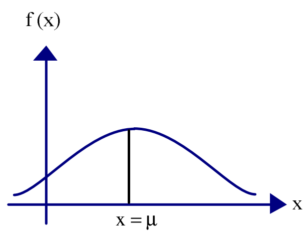

Fig. 2 X has normal distribution

H. The Exponential Distribution

A continuous random variable X assuming all nonnegative values is said to have an *exponential distribution* with parameter $\alpha > 0$ if its pdf given by

$$f(x) = \alpha e^{-\alpha x}, x > 0$$
$$= 0, \qquad (7)$$

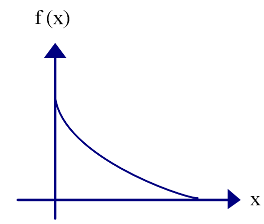

Fig. 3 X has exponential distribution

I.The Gamma Distribution

The Gamma Distribution, denoted by τ, is defined as follows:

$$\tau(p) = \int_0^\infty x^{p-1}e^{-x}dx, \text{ defined for } p > 0. \qquad (8)$$

Let X be a continuous random variable assuming only non-negative values. Then X has a *Gamma probability distribution* if its pdf is given by

$$f(x) = \frac{\alpha}{\tau(r)}(\alpha x)^{r-1}e^{-\alpha x},\ x > 0$$
$$= 0 \qquad \text{, elsewhere} \qquad (9)$$

This distribution depends on two parameters, r and α, of with $r > 0$ and $\alpha > 0$.

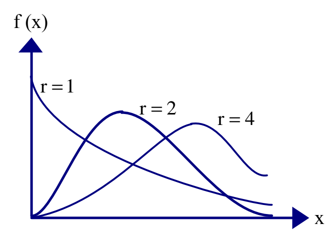

Fig. 4 X has Gamma distribution

III.THE MOMENT-GENERATING FUNCTIONS

A.Definitions

Let X be a discrete random variable with probability distribution $p(x_i) = P(X = x_i), i = 1,2,\ldots$. The Function M_X, called the *moment-generating function* of X, is defined by

$$M_X(t) = \sum_{j=1}^{\infty} e^{tx_j}p(x_j). \qquad (10)$$

If X is a continuous random variable with pdf f, the *moment-generating function* is defined by

$$M_X(t)=\int_{-\infty}^{+\infty} e^{tx} f(x)\,dx. \qquad (11)$$

In either the discrete or the continuous case, $M_X(t)$ is simply the expected value of e^{tX}.

$$M_X(t)=E\left(e^{tx}\right). \qquad (12)$$

$M_X(t)$ is the value which the function M_X assumes for the real variable t.

B. Example

Suppose that X is uniformly distributed over the interval $[a,b]$. Therefore the moment-generating function is given by

$$M_X(t)=\int_a^b \frac{e^{tx}}{b-a}dx$$

$$=\frac{1}{b-a}\left[e^{bt}-e^{at}\right],\ t\neq 0. \qquad (13)$$

C. Example

Suppose that X is binomially distributed with parameters n and p. Then

$$M_X(t)=\sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(pe^t\right)^k(1-p)^{n-k}$$

$$=\left[p^{e^t}+(1-p)\right]^n. \qquad (14)$$

D. Example

Suppose that X has a Poisson distribution with parameter λ. Thus

$$M_X(t)=\sum_{k=0}^{\infty} e^{tk}\frac{e^{-\lambda}\lambda^k}{k!}$$

$$=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\left(\lambda e^t\right)^k}{k!}$$

$$=e^{-\lambda}e^{\lambda e^t}=e^{\lambda(e^t-1)} \qquad (15)$$

E. Example

Suppose that X has an exponential distribution with parameter α. Therefore

$$M_X(t)=\int_0^{\infty} e^{tx}\alpha e^{-\alpha x}dx$$

$$=\alpha\int_0^{\infty} e^{x(t-\alpha)}dx$$

$$=\frac{\alpha}{\alpha-t},\quad t<\alpha. \qquad (16)$$

F. Example

Suppose that X has normal distribution $N\left(\mu,\sigma^2\right)$.

Hence $M_X(t)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx}\exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right)dx.$

Let $\frac{(x-\mu)}{\sigma}=s$; thus $x=\sigma s+\mu$ and $dx=\sigma ds$. Therefore

$$M_X(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[t(\sigma s+\mu)\right]e^{-\frac{s^2}{2}}ds$$

$$=e^{t\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left[s^2-2\sigma ts\right]\right)ds$$

$$=e^{t\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[(s-\sigma t)^2-\sigma^2t^2\right]\right\}ds$$

$$=e^{t\mu+\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}[s-\sigma t]^2\right)ds.$$

Let $s-\sigma t=\gamma$; then $ds=d\gamma$ and

$$M_X(t)=e^{t\mu+\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{\gamma^2}{2}}d\gamma$$

$$=e^{\left(t\mu+\frac{\sigma^2t^2}{2}\right)}. \qquad (17)$$

G. Example

Let X have a Gamma distribution with parameters α and r. Then

$$M_X(t) = \frac{\alpha}{\tau(r)}\int_0^\infty e^{tx}(\alpha x)^{r-1}e^{-\alpha x}dx$$

$$= \frac{\alpha^r}{\tau(r)}\int_0^\infty x^{r-1}e^{-x(\alpha-t)}dx.$$

Let $x(\alpha-t)=u$; thus

$$dx = \frac{(du)}{(\alpha-t)},$$

and

$$M_X(t) = \frac{\alpha^r}{(\alpha-t)\tau(r)}\int_0^\infty \left(\frac{u}{\alpha-t}\right)^{r-1}e^{-u}du$$

$$= \left(\frac{\alpha}{\alpha-t}\right)^r \frac{1}{\tau(r)}\int_0^\infty u^{r-1}e^{-u}du.$$

Since the integral equals $\tau(r)$,

$$M_X(t) = \left(\frac{\alpha}{\alpha-t}\right)^r. \quad (18)$$

If $r=1$, the Gamma function becomes the exponential distribution.

IV. PROPERTIES OF THE MOMENT-GENERATING FUNCTIONS

The Maclaurin series expansion of the function e^x;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$ Thus

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + ... + \frac{(tx)^n}{n!} + ...$$

Now

$$M_X(t) = E\left(e^{tX}\right) = E\left(1 + tX + \frac{(tX)^2}{2!} + ... + \frac{(tX)^n}{n!} + ...\right)$$

$$= 1 + tE(X) + \frac{t^2E\left(X^2\right)}{2!} + ... + \frac{t^nE\left(X^n\right)}{n!} + ...$$

Since M_X is a function of the real variable t, the derivative of $M_X(t)$ with respect to t, that is, $M'(t)$.

$$M'_X(t). = E(X) + tE\left(X^2\right) + \frac{t^2E\left(X^3\right)}{2!} + ... + \frac{t^{n-1}E\left(X^n\right)}{(n-1)!} + ...$$

Setting $t=0$,

$$M'(0) = E(X).$$

Thus the first derivative of the moment-generating function evaluated at $t=0$ yields the expected value of the random variable [6]. The second derivative of $M_X(t)$ is

$$M''(t) = E\left(X^2\right) + tE\left(X^3\right) + ... + \frac{t^{n-2}E\left(X^n\right)}{(n-2)!} + ...,$$

and setting $t=0$,

$$M''(0) = E\left(X^2\right).$$

The n^{th} derivative of $M_X(t)$ evaluated at t=0 is

$$M^{(n)}(0) = E\left(X^n\right).$$

The number $E\left(X^n\right), n=1,2,...,$ are called the n^{th} moments of the random variable X about zero.

The general Maclaurin series expansion of the function M_X is

$$M_X(t). = M_X(0) + M'_X(0)t + ... + \frac{M_X^{(n)}t^n}{n!} + ...$$

$$= 1 + \mu_1 t + \frac{\mu_2 t^2}{2!} + ... + \frac{\mu_n t^n}{n!} + ...$$

where $\mu_i = E\left(X^i\right), i=1,2,...$. In particular,

$$V(X) = E\left(X^2\right) - (E(X))^2$$

$$= M''(0) - \left[M'(0)\right]^2.$$

A. Theorem

Suppose that the random variable X has M_X. Let $Y = \alpha X + \beta$. Then M_Y, the moment-generator function of the random variable Y, is given by

$$M_Y(t) = e^{\beta t}M_X(\alpha t). \quad (20)$$

Proof:

$$M_Y(t)=E\left(e^{Yt}\right)=E\left[e^{(\alpha X+\beta)t}\right]$$
$$=e^{\beta t}E\left(e^{\alpha tX}\right)$$
$$=e^{\beta t}M_X(\alpha t).$$

B. Theorem

Suppose that X and Y are independent random variables. Let $Z=X+Y$. Let $M_X(t), M_Y(t)$ and $M_Z(t)$ be the moment-generating functions of the random variables X, Y and Z, respectively. Then

$$M_Z(t)=M_X(t)M_Y(t). \quad (21)$$

Proof:

$$M_Z(t)=E\left(e^{2t}\right)$$
$$=E\left[e^{(X+Y)t}\right]$$
$$=E\left(e^{Xt}e^{Yt}\right)$$
$$=E\left(e^{Xt}\right)E\left(e^{Yt}\right)=M_X(t)M_Y(t).$$

V. REPRODUCTIVE PROPERTIES OF DISTRIBUTIONS

If two or more independent random variables having a certain distribution are added, the resulting random variable has distribution of the same type as that of the summands. This result is the reproductive property[6].

A.Example

Suppose that X and Y are independent random variables with distributions $N\left(\mu_1,\sigma_1^2\right)$ and $N\left(\mu_2,\sigma_2^2\right)$, respectively. Let $Z=X+Y$. Hence

$$M_Z(t)=M_X(t)M_Y(t)=\exp\left(\mu_1 t,\frac{\sigma_1^2 t^2}{2}\right)\exp\left(\mu_2 t,\frac{\sigma_2^2 t^2}{2}\right)$$
$$=\exp\left((\mu_1+\mu_2)t,\left(\sigma_1^2+\sigma_2^2\right)\frac{t^2}{2}\right).$$

Thus Z has this normal distribution.

B. Example

The length of a rod is a normally distributed random variable with mean 4 inches and variance 0.01 inch^2. Two such rods are placed end to end and fitted into a slot. The length of this slot is 8 inches with a tolerance of ± 0.01 inch. The probability that the two rods will fit can be evaluated.

Letting L_1 and L_2 represent the lengths of rod 1 and rod 2, thus $L=L_1+L_2$ is normally distributed with $E(L)=8$ and $V(L)=0.02$. Hence

$$P[7.9\le L\le 8.1]=P\left[\frac{7.9-8}{0.14}\le\frac{L-8}{0.14}\le\frac{8.1-8}{0.14}\right]$$
$$=\Phi(+0.714)-\Phi(-0.714)=0.526,$$

from the tables of the normal distribution [4].

C. Theorem

Let $X_1,\ldots,X_n$ be independent random variables. Suppose that X_i has a Poisson distribution with parameter α_i, $i=1,2,\ldots,n$. Let $Z=X_1+\ldots+X_n$. Then Z has a Poisson distribution with parameter

$$\alpha=\alpha_1+\ldots+\alpha_n.$$

Proof:

For the case of $n=2$:

$$M_{X_1}(t)=e^{\alpha_1\left(e^t-1\right)},\ M_{X_2}(t)=e^{\alpha_2\left(e^t-1\right)}.$$

Hence $M_Z(t)=e^{(\alpha_1+\alpha_2)\left(e^t-1\right)}$. This is the moment-generating function of a random variable with Poisson distribution having parameter $\alpha_1+\alpha_2$. By the mathematical induction, the theorem is proved.

D.Example

Suppose that the number of calls coming into a telephone exchange between 9 a.m. and 10 a.m., X_1, is a random variable with Poisson distribution with parameter 3. Similarly, the number of calls arriving between 10 a.m. and 11 a.m, X_2, also has a Poisson distribution, with parameter 5. If X_1 and X_2 are independent, the probability that more than 5 calls come in between 9 a.m. and 11 a.m. can be solved[3].

Let $Z = X_1 + X_2$. From the above theorem, Z has a Poisson distribution with parameter $3+5=8$. Hence

$$P(Z>5) = 1 - P(Z \le 5)$$
$$= 1 - \sum_{k=0}^{5} \frac{e^{-8}(8)^k}{k!}$$
$$= 1 - 0.1919 = 0.8088.$$

E. Theorem

Suppose that $X_1, \dots, X_k$ are independent random variables, each having distributions $N(0,1)$. Then

$$S = X_1^2, X_2^2 + \dots + X_k^2 \text{ has distribution } X_k^2.$$

F. Example

Suppose that $X_1, \dots, X_n$ are independent random variables, each with distribution $N(0,1)$. Let $T = \sqrt{X_1^2, \dots, X_n^2}$. Since T^2 has distribution X_n^2.

$$H(t) = P(T \le t) = P\left(T^2 \le t^2\right)$$
$$H(t) = \int_0^{t^2} \frac{1}{2^{\frac{n}{2}} \tau\left(\frac{n}{2}\right)} z^{\frac{n}{2}-1} e^{-\frac{z}{2}} dz.$$

Hence

$$h(t) = H'(t)$$
$$= \frac{2t}{2^{\frac{n}{2}} \tau\left(\frac{n}{2}\right)} \left(t^2\right)^{\frac{n}{2}-1} e^{-\frac{t^2}{2}}$$
$$= \frac{2t^{n-1} e^{-\frac{t^2}{2}}}{2^{\frac{n}{2}} \tau\left(\frac{n}{2}\right)} \text{ if } t \ge 0.$$

V. CONCLUSIONS

The moment-generating function as defined above is written as an infinite series or improper integral depending on whether the random variable is discrete or continuous. The method of moment-generating functions to evaluate the expectation and variance of a random variable with probability distribution are used. And then we have discussed a number of distribution for which a reproductive property holds. We have seen that the moment-generating function can be a powerful tool for studying various aspects of probability distributions. We found the use of the moment-generating function very helpful in studying sums of independent, identically distributed random variables and obtaining various reproductive laws.

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