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Flexural Analysis of Thick Beams Using Trigonometric Shear Deformation Theory

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Abstract:

In this present study, A trigonometric shear deformation theory is developed for static flexural analysis of thick beams, in which number of variables is same as that in the first-order shear deformation theory.The sinusoidal function is used in displacement field in terms of thickness coordinate to represent the shear deformation effect. The Governing differential equation and boundary condition of the theory are obtained by using principle of virtual work. The fixed beam subjected to sine load is examined using present theory. The numerical results obtained are compared with those of Elementary, Timoshenko, and Higher-order shear deformation theory and with the available solution in the literature.

Keywords **—Trigonometric shear deformation theory, Transverse shear stress, Thick beam, Principle of virtual work, Equilibrium equation, axial shear stress.**

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I. INTRODUCTION

The classical beam theory (ETB) is based on Bernoulli-Euler hypothesis and the classical plate theory (CPT) is based on Kirchhoff's hypothesis. These theories are widely used for the analysis of thin beams and plates. These theories are not adequate for the analysis of shear flexible beams due to the neglect of transverse shear deformations in the theories, as a consequence they under predict deflections, and over predict natural frequencies and buckling loads. This led to the development shear deformation theories for thick beams and plates. The shear deformation theories are those in which the transverse shear deformation or transverse normal deformation effects are accounted for.

Theories of beams and plates are essentially one and two-dimensional approximations of the corresponding two and three-dimensional theories of elastic bodies i.e., beams and plates. These are basically the reduction problem. Since the thickness dimension is much less than the others, it is responsible to approximate the distribution of the displacements, strains and stress components in thickness dimension.

Itiswellknownthat elementary theory of bending of beam based on Euler-Bernoulli hypothesis that the plane sections which are perpendicular to the neutral axis before bending, implying that the transverse shear and transverse normal strains are zero. Thus, the theory disregards the effects of the shear deformation. It is also known as classical beam theory. The theory is applicable to slender beams and should not be applied to thick or deep beams.

The next theory is the hierarchy of beam theories is theTimoshenko [1]beam theory or first-order shear deformation theory (FSDT) is an improvement over the elementary theory of beam.In Timoshenko beam theory, the normality assumption of the Euler-Bernoulli beam theory is relaxed and a constant state of transverse shear strain with respect to the thickness coordinate is include.

In order to remove the deficiencies in ETB and FSDT, higher order shear deformation theories are developed to obtain the improve global response. In higher order theories, the Euler- Bernoulli hypothesis is further relaxed by removing the straightness assumption. In these theories the displacement field is expanded up to the third power of thickness coordinate of beams to have the parabolic variation of transverse shear stresses. The third order beam theory given byLevinson [2], Bickford [3], and Rehfield and Murty [4] which is based on the displacement field.Ghugal [5] has developed a beam theory including the transverse shear strain and non classical axial stress. There exists another method of reduction of the two and three-dimensional bending problems of elasticity theoryto the corresponding one and two-dimensional approximate problems of mechanics of elastic bodies. This can be accomplished by assuming a displacement field containing a sinusoidal function associated with shear deformation effect. Refined beam theories with the introduction of trigonometric function in terms of thickness coordinate in kinematical assumption are introduced by Vlasov and Leontev [6], stein [7]. Ghugal and Shimpi [8] has developed a variationally consistent refined trigonometric shear deformation theory for flexure and free vibration of beams. The number of displacement variables in this theory is three as compared to two in FSDT. The theory satisfies the zero transverse shear stress conditions on the top and bottom surfaces of the beam and thus obviates the need of shear correction factor.

In this paper, a trigonometric shear deformation theory is developed for static flexural analysis of thick beams. The theory is applied to a fixed beam to analyzed the axial displacement, Transverse displacement, axial displacement, axial bending stress and transverse shear stress. The numerical results have been computed for various length to thickness ratios of the beams and the results obtained are compared with those of Elementary, Timoshenko, trigonometric and other hyperbolic shear deformation theories and with the available solution in the literature.

2.Formulation of problem

The beam under consideration as shown in Figure1 occupies in 0-x-y-z Cartesian coordinate system the region:

$0 \le x \le L$; $- b/2 \le y \le b/2$; $- h/2 \le z \le h/2$

where x, y, z are Cartesian coordinates, L and b are the length and width in the x and y directions respectively, and h is the thickness of the beam. The beam is made up of homogeneous, linearly elastic isotropic material.

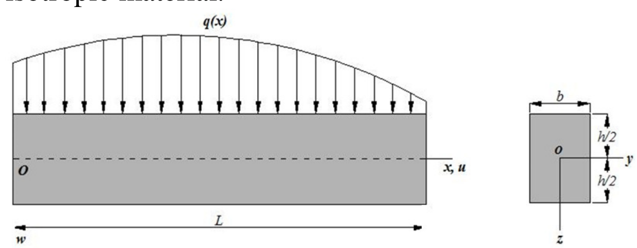

Figure1: Beam under bending in x-z plane

A. The displacement field

The displacement field of the present beam theory is of the form

$$u(x,z) = -z\frac{\partial w}{\partial x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x) \qquad (1)$$

$$w(x,z) = w(x)$$

Where,

u = Axial displacement in x direction which is function of x and z.

w = Transverse displacement in z direction which is function of x.

ϕ = Rotation of cross section of beam at neutral axis which is function of x.

Normal Strain:

$$\in_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \left[\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]\frac{\partial \phi}{\partial x} \qquad (2)$$

Shear strain:

$$\gamma_X = \left[\cos\left(\frac{\pi z}{h}\right)\right]\phi \quad (3)$$

Stresses:

$$\sigma_X = E\varepsilon_X = -zE\frac{\partial^2 w}{\partial x^2} + \left[E\frac{h}{\pi}\sin\frac{\pi z}{h}\right]\frac{\partial\phi}{\partial x} \quad (4)$$

$$\tau_X = G\left[\cos\left(\frac{\pi z}{h}\right)\right]\phi \quad (5)$$

where E and G after elastic constant of the beam material.

B. Governing differential equation

Governing differential equations and boundary conditions are obtained from Principle of virtual work. Using equations for stresses, strains and principle of virtual work, variationally consistent differential equations for beam under consideration are obtained. The principle of virtual work when applied to beam leads to:

$$b\int_{x=0}^{x=L}\int_{Z=-\frac{h}{2}}^{z=\frac{h}{2}}(\sigma_x\delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz})dxdz - \int_{x=0}^{x=L} q\delta w dx = 0 \quad (6)$$

Where δ = variational operator

Employing Greens theorem equation successively we obtain the coupled Euler Lagrange's equation which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations are obtained areas follows.

$$EI\frac{\partial^4 w}{\partial x^4} - A_0\frac{\partial^3\phi}{\partial x^3} = q(x) \quad (7)$$

$$EI\left(A_0\frac{\partial^3 w}{\partial x^3} - B_0\frac{\partial^2\phi}{\partial x^2}\right) + GAC_0\phi \quad (8)$$

Where A_0, B_0 and C_0 are the stiffness coefficients in governing equations. The associated consistent natural boundary conditions obtained are of following form along the edges x = 0 and x = L.

$$V_x = EI\left(\frac{\partial^3 w}{\partial x^3} - A_0\frac{\partial^2\phi}{\partial x^2}\right) = 0 \quad (9)$$

Where w is prescribed

$$M_x = EI\left(\frac{\partial^2 w}{\partial x^2} - A_0\frac{\partial\phi}{\partial x}\right) = 0 \quad (10)$$

Where $\frac{dw}{dx}$ is prescribed.

$$M_x = EI\left(A_0\frac{\partial^2 w}{\partial x^2} - B_0\frac{\partial\phi}{\partial x}\right) = 0 \quad (11)$$

Where ϕ is Prescribed.

C. The General solution of Governing equilibrium equations of beam:

The General solution for transverse displacement w(x) and ϕ(x) can be obtained from eqn. (7) and (8) by discarding time (t) derivatives. Integrating and rearranging the eqn. (7), we obtained the following equation,

$$\frac{\partial^3 w}{\partial x^3} = A_0\frac{\partial^2\phi}{\partial x^2} + \frac{Q(x)}{D} \quad (12)$$

where, ϕ(x) is generalized shear force for beam and it is given by,

$$\frac{\partial^3 w}{\partial x^3} = \frac{B_0}{A_0}\frac{\partial^2\phi}{\partial x^2} - \beta\phi \quad (13)$$

And by rearranging second governing Eqn. (8) the following equation is obtained.

$$Q(x) = \int_0^x q dx + C_1 \quad (14)$$

Now a single equation in terms of ϕ is obtained, by putting the Eqn. (8) in second governing Eqn. (14)

$$\alpha\frac{\partial^2\phi}{\partial x^2} - \beta(\phi) = \frac{Q(x)}{EI} \quad (15)$$

$$\phi = C_2 \cosh(\lambda x) + C_3 \sinh(\lambda x) - \left[\frac{Q(x)}{\beta EI}\right] \quad (16)$$

The equation of transverse displacement w(x) is obtained by substituting the expression of $\phi(x)$ in Eqn. (15) and integrating it thrice with respect to x. The general solution for w(x) is obtained as follows:

$$EIw(x) = \iiiint q\,dx\,dy\,dz + \frac{C_1 x^3}{6} + \left(\frac{\pi}{4}\lambda^2 - \beta\right)\frac{EI}{\lambda^3}\left(C_2 \sinh\lambda x + C_3 \cosh\lambda x\right) + \frac{C_4 x^2}{2} + C_5 x + C_6 \quad (17)$$

where C1, C2, C3, C4, C5 and C6 are the constants of integration and can be obtained by imposing natural (forced) and kinematic boundary conditions of beams.

3. Illustrative Example

In order to prove the efficiency of the present theory, the following numerical example are considered. The following material properties for beam are used. Material properties:

1. Modulus of Elasticity E = 210 GPa

2. Poisson's ratio μ = 0.30

3. Density = 7800 kg/m3

A. Fixed beam subjected to sine loading, $q(x) = q_0 \sin\frac{\pi x}{L}$

The beam has its origin on left hand side support and is fixed at $x = 0$ and $x = L$. The beam is subjected to sine loading, on surface z = +h/2 acting in downward z direction with minimum intensity minimum intensity of load q0.

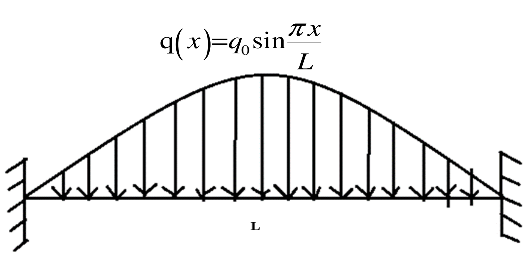

Fig.2: Fixed beam subjected to sine load

Boundary conditions associated with this problem are as follows:

At fixed end: x=0, L

$$\frac{\partial w}{\partial x} = \phi = w = 0$$

General Expressions obtained for $w(x)$ and $\phi(x)$ are as follows.

$$w(x) = \left[\begin{array}{l} \left\{\frac{120}{\pi^3}\left(\frac{1}{\pi}\sin\frac{\pi x}{L} - \frac{x}{L}\right) + 10\frac{E}{G}\frac{h^2}{L^2}\frac{B_0}{C_0}\frac{1}{\pi}\left(\frac{1}{\pi}\sin\frac{\pi x}{L} - \frac{x}{L} + \frac{x^2}{L^2}\right)\right. \\ \left. + 10\frac{E}{G}\frac{h^2}{L^2}\frac{A_0^2}{C_0}\left(\frac{\cosh\lambda x - \sinh\lambda x - 1}{\lambda L} + \frac{x}{L} - \frac{1}{2}\frac{x^2}{L^2}\right)\right\} \end{array}\right] \quad (18)$$

$$\phi(x) = \frac{A_0}{C_0}\frac{1}{\pi}\frac{q_0 L}{Gbh}\left[\sinh\lambda x - \cosh\lambda x + \cos\frac{\pi x}{L}\right] \quad (19)$$

Expression for Non-DimensionalAxial Displacement $(\overline{u})$, Transverse Deflection $(\overline{w})$, Axial Stress $(\overline{\sigma_x})$, Maximum Transverse Shear Stresses $(\overline{\tau_{zx}^{CR}})$ and $(\overline{\tau_{zx}^{EE}})$ are as follows.

$$\bar{u} = -\frac{z}{h}\frac{L^3}{h^3}\left[\begin{array}{l} \left\{\frac{12}{\pi^3}\left(\cos\frac{\pi x}{L} - 1 + \frac{2x}{L}\right)\right. \\ \left. + \frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{1}{\pi}\left(\cos\frac{\pi x}{L} - 1 + \frac{2x}{L}\right)\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{1}{\pi}\left(\sinh\lambda x - \cosh\lambda x + 1 - \frac{x}{L}\right)\right\} \\ + \left[\frac{h}{\pi}\sin\frac{\pi z}{h}\right]\frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{1}{\pi}\left(\sinh\lambda x - \cosh\lambda x + \cos\frac{\pi x}{L}\right) \end{array}\right] \quad (20)$$

$$\overline{\sigma_x} = -\frac{z}{h}\frac{L^2}{h^2}\left[\begin{array}{l} \left\{\frac{12}{\pi^3}\left(-\sin\frac{\pi x}{L}\pi + 2\right) + \frac{E}{G}\frac{A_0^2}{C_0}\frac{h^2}{L^2}\frac{1}{\pi}\left(\lambda L\cosh\lambda x - \lambda L\sinh\lambda x - 1\right)\right. \\ \left. + \frac{E}{G}\frac{B_0}{C_0}\frac{h^2}{L^2}\frac{1}{\pi}\left(-\sin\frac{\pi x}{L}\pi + 2\right)\right\} \\ + \left[\frac{h}{\pi}\sin\frac{\pi z}{h}\right]\frac{E}{G}\frac{1}{\pi}\frac{A_0^2}{C_0}\left(\lambda L\cosh x - \lambda L\sinh\lambda x - \pi\sin\frac{\pi x}{L}\right) \end{array}\right] \quad (21)$$

$$\overline{\tau_{zx}^{CR}} = \left[\frac{A_0}{C_0}\frac{L}{h}\frac{1}{\pi}\left(\cos\frac{\pi z}{h}\right)\left(\sinh\lambda x - \cosh\lambda x + \cos\frac{\pi x}{L}\right)\right]$$

(22)

$$\overline{\tau_{zx}^{EE}} = \frac{1}{8}\frac{L}{h}\left(\frac{4z^2}{h^2} - 1\right)\left[\begin{array}{l} \left\{\frac{12}{\pi}\left[-\cos\frac{\pi x}{L}\right] - \pi\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left[\cos\frac{\pi x}{L}\right]\right. \\ \left. + \frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{1}{\pi}\left(\lambda^2 L^2\sinh\lambda x - \lambda^2 L^2\cosh\lambda x\right)\right\} \\ + \frac{A_0}{C_0}\frac{E}{G}\frac{1}{\pi}\frac{h}{L}\left[\frac{h}{\pi}\cos\frac{\pi}{2}\right]\left(\lambda^2 L^2\sinh\lambda x - \lambda^2 L^2\cosh\lambda x - \pi^2\cos\frac{\pi x}{L}\right) \end{array}\right] \quad (23)$$

4. Numerical Result

The numerical results for axial displacements, transverse displacements, bending stress and transverse shear stress are presented in following non dimensional form and the values are presented in table 1 and table 2.

$$\bar{w}=\frac{10Ebh^3}{q_0L^4}w \quad \bar{u}=\frac{Eb}{q_0h}u$$

$$\bar{\sigma}_x=\frac{b}{q_0}\sigma_x \qquad \bar{\tau}_{zx}=\frac{b}{q_0}\tau_{zx}$$

Table 1: Non-Dimensional Axial Displacement ($\bar{u}$) at (x =0.75L, z = h/2), Transverse Deflection ($\bar{w}$) at (x = 0.75L, z = 0.0), Axial Stress ($\bar{\sigma_x}$) at (x = 0, z = h/2), Maximum Transverse Shear Stresses ($\overline{\tau_{zx}^{CR}}$) and ($\overline{\tau_{zx}^{EE}}$) at (x = 0:01L, z = h/2) of Fixed Beam Subjected to Varying Load for Aspect Ratio 4

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	TSDT	2.12	1.76	-1.36	-2.207	474.01
Dahake	HSDT	2.13	2.21	-1.36	-0.029	650.42
Timoshenko	FSDT	1.44	6.44	-1.34	-2.903	4.77
Bernoulli-Euler	ETB	0.14	6.44	-1.34	-	4.77

Table 2: Non-Dimensional Axial Displacement ($\bar{u}$) at (x =0.75 L, z = h/2), Transverse Deflection ($\bar{w}$) at (x =0.75 L, z = 0.0), Axial Stress ($\bar{\sigma_x}$) at (x = 0, z = h/2), Maximum Transverse Shear Stresses ($\overline{\tau_{zx}^{CR}}$) and ($\overline{\tau_{zx}^{EE}}$) at (x = 0:01L, z = h/2) of Fixed Beam Subjected to Varying Load for Aspect Ratio 10.

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	TSDT	2.12	1.76	-1.36	-2.207	474.01
Dahake	HSDT	2.13	2.21	-1.36	-0.029	650.42
Timoshenko	FSDT	1.44	6.44	-1.34	-2.903	4.77
Euler-Bernoulli	ETB	0.14	6.44	-1.34	-	4.77

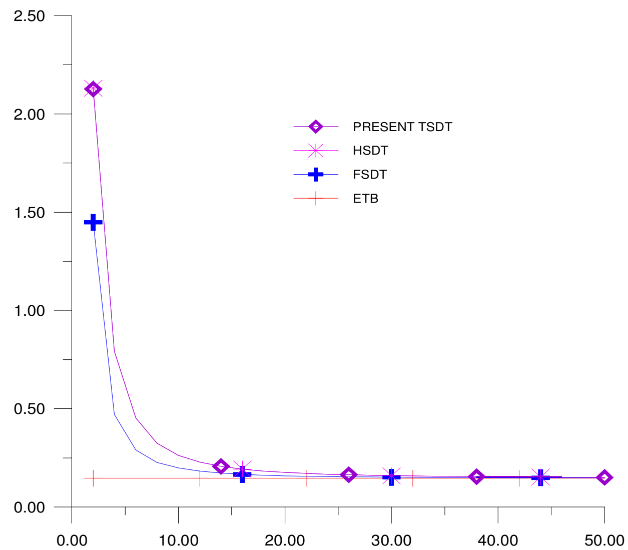

Fig 3: Variation of transverse displacements $\bar{w}$

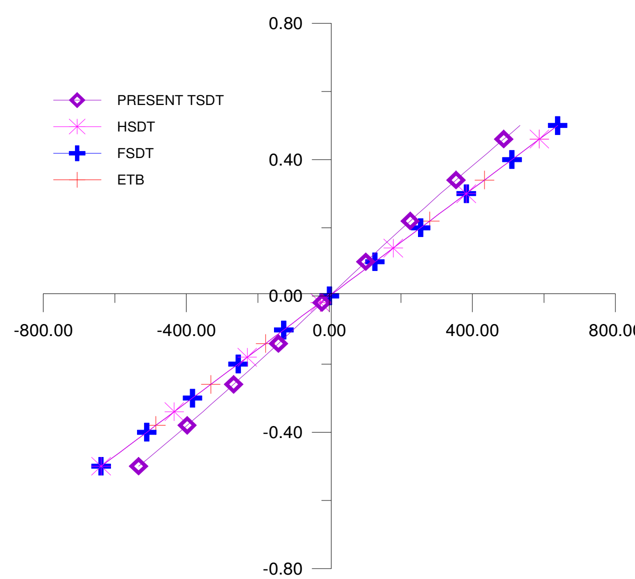

Fig4: Variation of Maximum Axial displacement $\bar{u}$u for AS 04

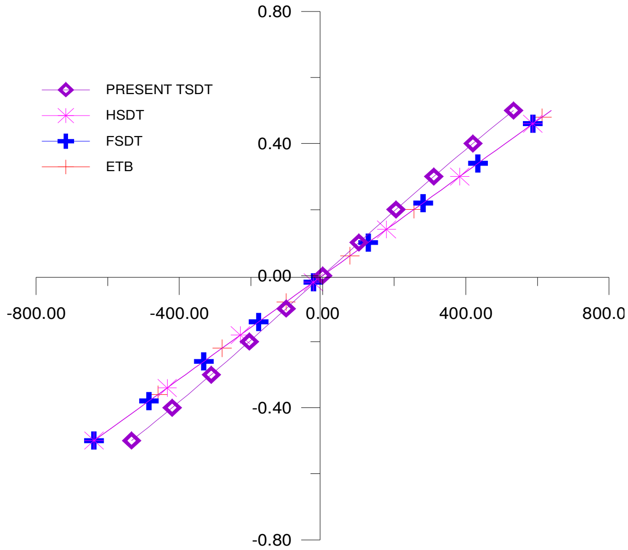

Fig 5: Variation of Maximum Axial displacement u for AS 10

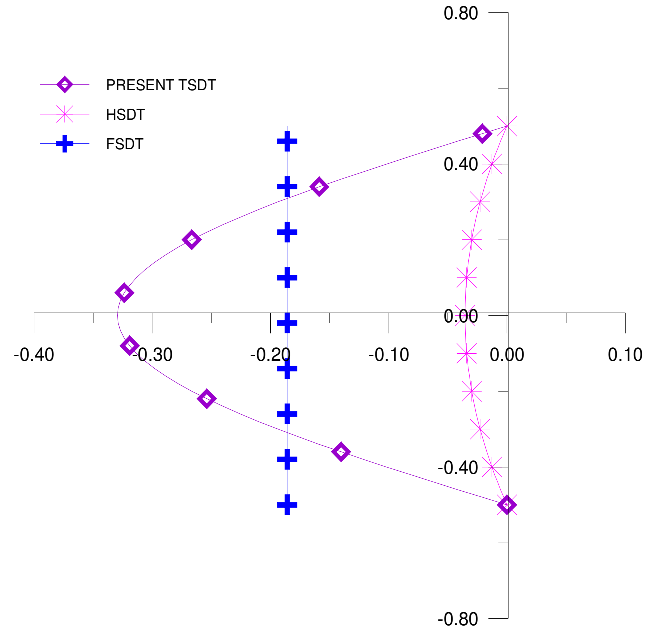

Fig 6: Variation of trans verse shear stress $\overline{\tau_{zx}^{CR}}$ for AS 04

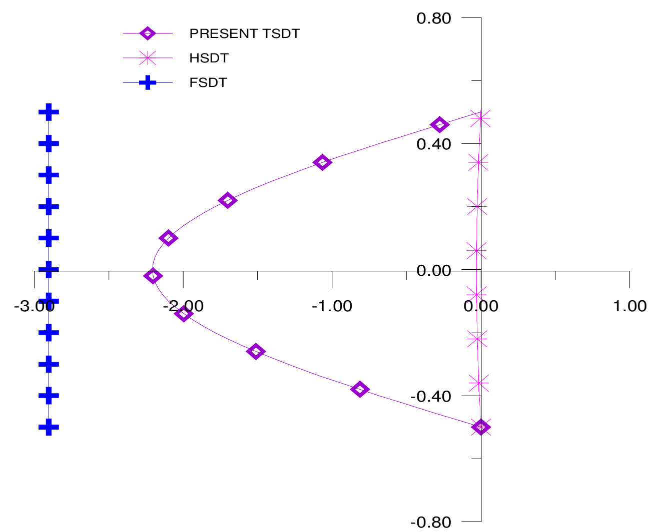

Fig 7: Variation of transverse shear stress $\overline{\tau_{zx}^{CR}}$ for AS 10

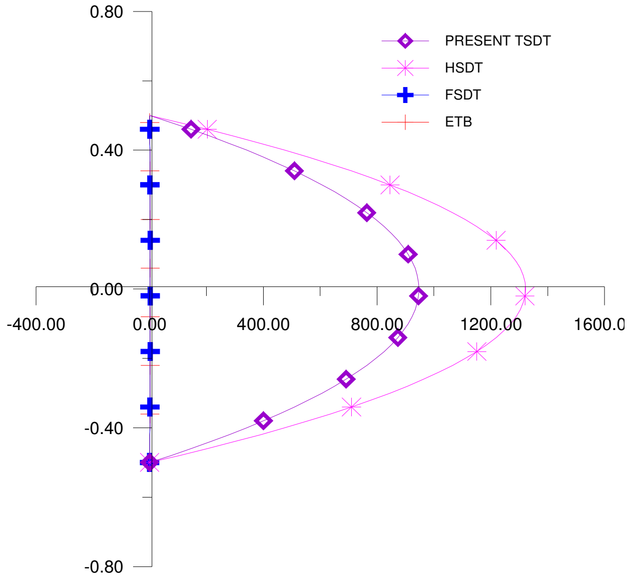

Fig 8: Variation of transverse shear stress $\overline{\tau_{zx}^{EE}}$ for AS 04

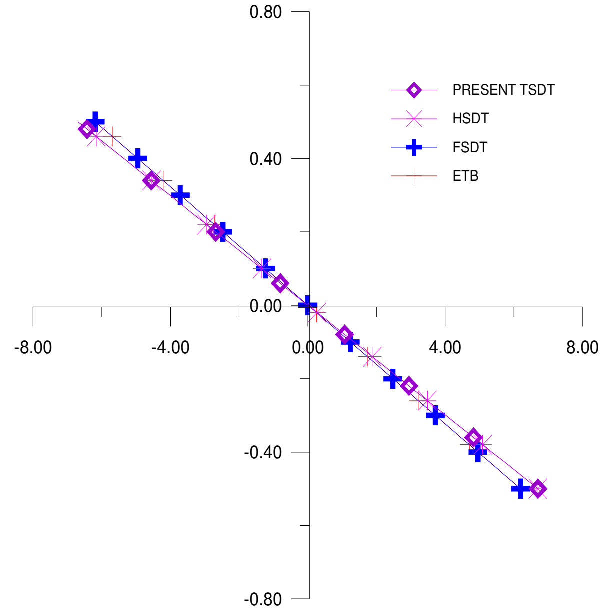

Fig 10: Variation of maximum axial stress $\overline{\sigma_x}$ For AS 04

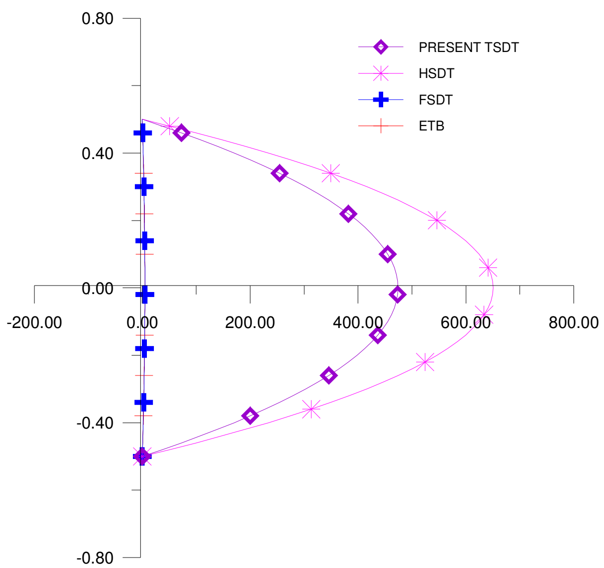

Fig 9: Variation of transverse shear stress $\overline{\tau_{zx}^{EE}}$ for AS 10

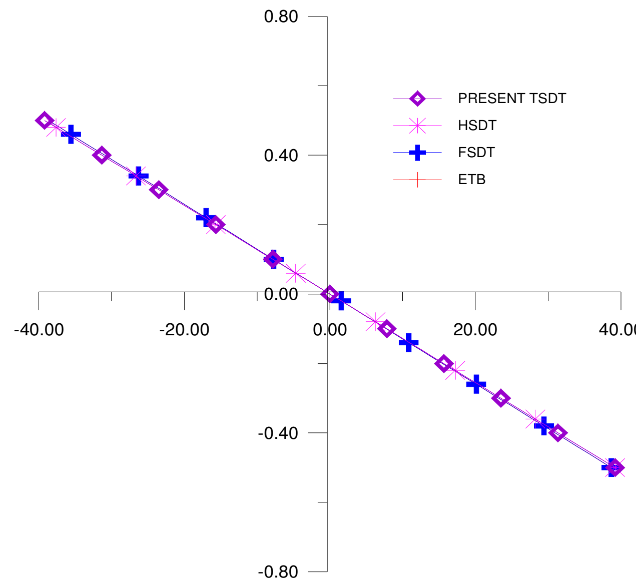

Fig 11: Variation of maximum axial stress $\overline{\sigma_x}$ For AS 10

5. CONCLUSIONS

From the static flexural analysis of fixed beam following conclusions are drawn:

1. The result of maximum transverse displacement $\overline{w}$ obtained by present theory is in excellent agreement with those of other equivalent refined theories. The variation of AS 04 and AS 10 are present as shown in fig. 3

2. From figure 4 and 5, it can be observed that, the result of axial displacement $\overline{u}$ for beam subjected to uniformly load varies linearly through the thickness of beam of AS 04 and AS 10 respectively.

3. The transverse shear stress $\overline{\tau_{zx}^{CR}}$ and $\overline{\tau_{zx}^{EE}}$ are obtained directly by constitutive relation. Fig. 6, 7, 8, and 9. Shows the through thickness variation of transverse shear stress for thick beam for AS 04 and AS 10. From this fig. it can be observed that, the transverse shear stress satisfies the zero condition at top and at bottom surface of the beam.

4. The maximum Non-dimensional axial stress $\overline{\sigma_x}$ For AS 04 and AS 10 varies linearly through the thickness of beam as shown in fig.10 and fig. 11 respectively.

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