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Statistical Properties of the Exponentiated Alpha Power Inverted Exponential distribution and its Application

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Abstract:

The primary purpose of this research is to develop a model that can be used to simulate the UK COVID-19 mortality rate data. In this case, the Exponentiated Alpha Power Inverted Exponential distribution, abbreviated as EAPIEx distribution, is used with two shapes and one scale parameter(s).
The statistical properties of the new distribution are derived and investigated, and the method of maximum likelihood estimation was used to estimate the unknown parameters of the new distribution. The estimator's effectiveness is assessed, and the Monte Carlo simulation results of the estimates are reported in tabular form. The EAPIEx distribution was utilized to analyze the flexibility and adaptability of the distribution using UK COVID-19 Mortality Rate Data, and it turned out to perform better than the other models used in this study.

Keywords: Statistical Properties, Exponentiated Alpha Power, Inverted Exponential, Maximum Likelihood estimation

I. Introduction

The probability density function (pdf) is a key concept in probability theory and statistics that allows us to understand and assess continuous random variables and the probability distributions that go with them. It provides a mathematical representation of the relative likelihood of multiple options, allowing for statistical calculations and modeling procedures.

Exponentiated functions are used extensively in a variety of disciplines, including economics, physics, biology, and computer science. They give mathematicians a way to model and examine a variety of exponentially growing or decaying processes.
Since 1995, the exponentiated distributions have received much statistical study, and several authors have created several classes of these distributions by [1]. The Exponentiated Generalized Class of distributions by [2], which expands the concept first put forth by [3] and researched by [4] by adding two parameters to a continuous distribution. [5] first presented the Exponentiated Weibull distribution, and several authors have also created and generalized a number of standard distributions based on these distributions. The Exponentiated Inverted Exponential Distribution by [6] determines that the proposed model is positively skewed and its shape could be decreasing or unimodal depending on its parameter values. A new versatile modification of the Rayleigh distribution for modeling COVID-19 mortality rate by [7], uses the Exponentiated Generalized family to model COVID-19 mortality rate data. On The Exponentiated Generalized Exponentiated Exponential Distribution with Properties and Application by [8], they used the Exponentiated Generalized family as well to model Fatigue life of 6061-T6 aluminum coupons. Exponentiated Inverse Rayleigh Distribution and an Application to Coating Weights of Iron Sheets Data by [9].

A class of transformations called the Alpha Power Transform is employed in data analysis. This method aids researchers in enhancing or normalizing the distributional characteristics of the data they are working with.
The Alpha Power Transformation Family: Properties and Applications by [10]. Alpha-Power Exponentiated Inverse Rayleigh distribution and its application to real and simulated data by [11]. Alpha Power Transformed Weibull-G Family of Distributions: Theory and Applications by [12]. The Alpha Power Exponentiated Inverse Exponential distribution and its application to Italy's COVID-19 mortality rate data by [13], compared the new distribution with other distributions and it outperformed the other distributions.

The cdf and pdf of the exponentiated family of distributions are provided as follows, if Y is a random variable, the Exponentiated Alpha Power family distribution is defined by exponentiating the Alpha Power transform;

$$G_{EAPF}(y) = \left(\frac{\alpha^{W(y)} - 1}{\alpha - 1}\right)^{\mu} \tag{1}$$

the pdf corresponding to the cdf is defined as;

$$g_{EAPF}(y) = \frac{\mu \log \alpha}{(\alpha - 1)^{\mu}} w(y) \alpha^{W(y)} \left(\alpha^{W(y)} - 1\right)^{\mu - 1} \tag{2}$$

$\forall y, \alpha, \mu > 0$, $\alpha \neq 1$, and α, μ are shape parameters.

The inverted exponential distribution, sometimes referred to as the Pareto Type I distribution or the inverse exponential probability distribution, represents continuous random variables with non-negative support.
Theoretical Analysis of the Kumaraswamy- Inverse Exponential Distribution by [14]. The Transmuted Inverse Exponential Distribution by [15]. A novel extended inverse-exponential distribution and its application to COVID-19 data by [16].
The inverted exponential (IEx) distribution is defined by [17] with probability density function (pdf) and cumulative distribution function (cdf) respectively as;

$$w(y) = \frac{\beta}{y^2} e^{-\beta y^{-1}} \tag{3}$$

$$W(y) = e^{-\beta y^{-1}} \tag{4}$$

$\forall y, \beta > 0$ and β is a scale parameter

II. The EAPIEx distribution

The three-parameter EAPIEx distribution is derived by utilizing eqs.(3) & (4) as the baseline distributions.

By utilizing the pdf and cdf of the baseline distributions, respectively, the pdf and cdf of the EAPIEx distribution in eqs.(1) & (2), can be rewritten as;

$$g(y) = \frac{\mu \beta \log \alpha}{y^2 (\alpha - 1)^{\mu}} e^{-\beta y^{-1}} \alpha^{e^{-\beta y^{-1}}} \left(\alpha^{e^{-\beta y^{-1}}} - 1\right)^{\mu - 1} \tag{5}$$

$$G(y) = \left(\frac{\alpha^{e^{-\beta y^{-1}}} - 1}{\alpha - 1}\right)^{\mu} \tag{6}$$

$\forall y, \alpha, \mu, \beta > 0$, $\alpha \neq 1$, α, μ are shape parameters, and β is a scale parameter

A. Linear presentation of the pdf and cdf of the EAPIEx distribution

The pdf and cdf of the EAPIEx distribution can be presented linearly by utilizing some mathematical techniques on eqs.(5) & (6).

a. The pdf of the EAPIEx distribution is written linearly as;

$$g(y) = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu - 1}{j} \frac{(\log \alpha)^{i+1}}{i!} y^{-2} \left(e^{-\beta y^{-1}}\right)^{i+1} \times \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - 1 - j} \tag{7}$$

Proof:

From series notation $\alpha^x = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} x^i$ and binomial expansion $(m - n)^{p-1} = \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} m^{p-1-j} n^j$

From eq.(5), the following expressions are simplified as;

$$\alpha^{e^{-\beta y^{-1}}} = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \left(e^{-\beta y^{-1}}\right)^i$$

and

$$\left(\alpha^{e^{-\beta y^{-1}}} - 1\right)^{\mu - 1} = \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu - 1}{j} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - 1 - j}$$

Hence eq.(5) can be rewritten as;

$$g(y) = \frac{\mu \beta \log \alpha}{y^2 (\alpha - 1)^{\mu}} e^{-\beta y^{-1}} \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \left(e^{-\beta y^{-1}}\right)^i \times \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu - 1}{j} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - 1 - j} \tag{8}$$

eq.(8) can further be simplified as;

$$g(y) = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu - 1}{j} \frac{(\log \alpha)^{i+1}}{i!} y^{-2} \left(e^{-\beta y^{-1}}\right)^{i+1} \times \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - 1 - j} \tag{9}$$

where; $\Phi = \mu \beta (\alpha - 1)^{-\mu}$

b. The cdf of the EAPIEx distribution is written linearly as;

$$G(y) = (\alpha - 1)^{-\mu} \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - v} \tag{10}$$

Proof:

From eq.(6) the following expression is simplified as;

$$\left(\alpha^{e^{-\beta y^{-1}}} - 1\right)^{\mu} = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - v}$$

Hence eq.(6) can be rewritten as;

$$G(y) = (\alpha - 1)^{-\mu} \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - v} \tag{11}$$

The Survival (Reliability) and Hazard (Failure rate) functions corresponding to the EAPIEx distribution are given respectively as;

$$\begin{aligned} S(y) &= 1 - G(y) \\ &= 1 - (\alpha - 1)^{-\mu} \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu - v} \end{aligned} \tag{12}$$

and

$$\begin{aligned} H(y) &= \frac{g(y)}{S(y)} \\ &= \frac{\mu\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log\alpha)^{i+1}}{i!} \times y^{-2}\left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}}{(\alpha-1)^{\mu} - \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-v}} \end{aligned} \tag{13}$$

$\forall y, \alpha, \mu, \beta > 0,\ \alpha \neq 1$

B. EAPIEx distribution Submodel

The EAPIEx distribution has a well-known submodel when $\mu = 1$, the distribution reduces to Alpha Power Inverted Exponential (APIEx) distribution by [18].

The APIEx distribution is written as;

$$\text{cdf;}\ \ G(y) = \frac{\alpha^{e^{-\beta y^{-1}}} - 1}{\alpha - 1} \tag{14}$$

and its corresponding pdf is;

$$g(y) = \frac{\beta \log \alpha}{y^2(\alpha - 1)} e^{-\beta y^{-1}} \alpha^{e^{-\beta y^{-1}}} \tag{15}$$

$\forall y, \alpha, \beta > 0,\ \alpha \neq 1$, α is a shape parameter, and β is a scale parameter

For the parameter values selected, the pdf plots in Figure 1 display a positively skewed, J-shape, reversed J-shape, increase, and unimodal.

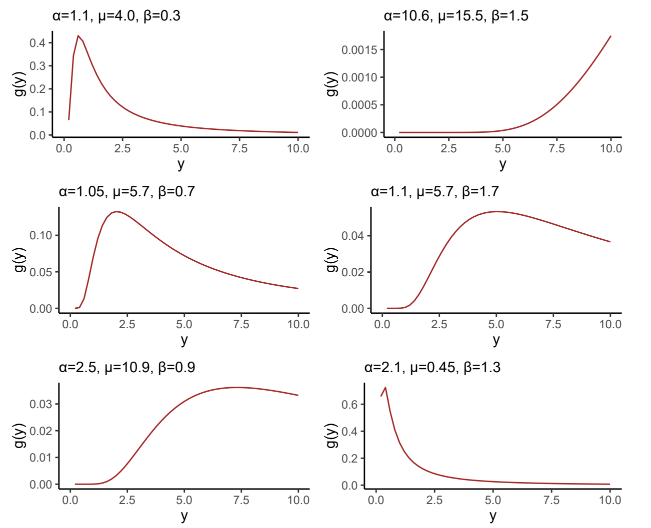


Figure 1: The pdf plots of the EAPIEx distribution

The $S(y)$ plots in Figure 2 show a monotonic decrease or a non-monotonic increase shape.

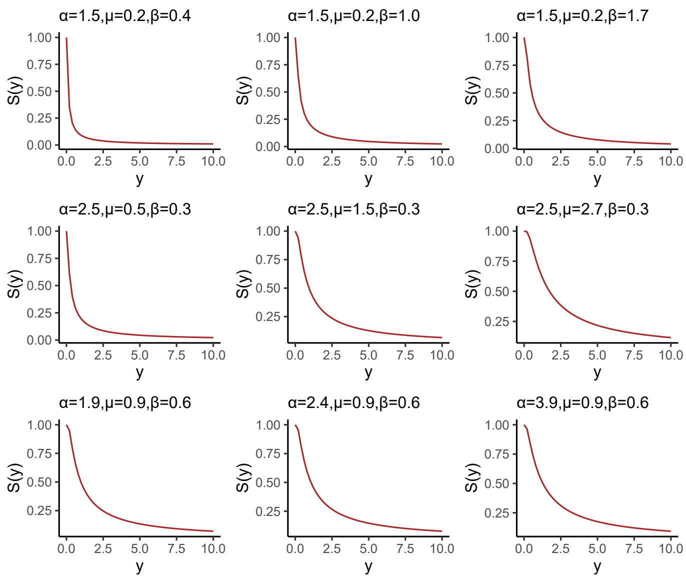


Figure 2: The S(y) plots of the EAPIEx distribution

The $H(y)$ plots in Figure 3, show a decreasing, an increasing, and an inverted bathtub shape.

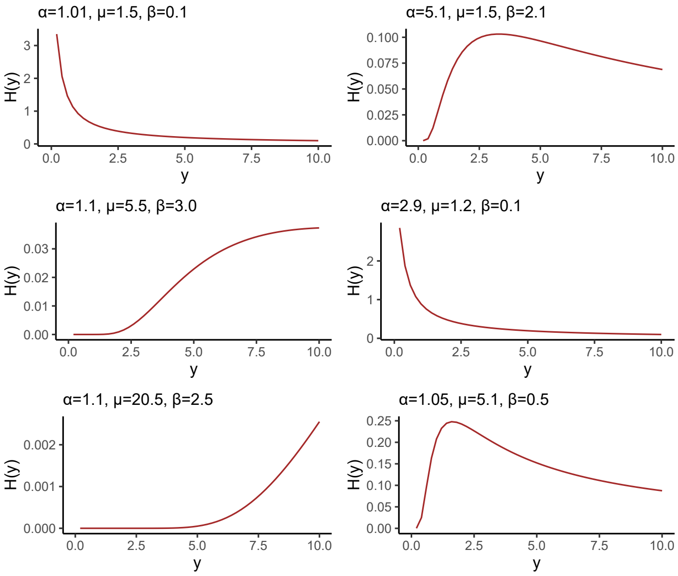


Figure 3: The H(y) plots of the EAPIEx distribution

III. Statistical Properties

The statistical properties of the EAPIEx distribution, which includes the quantile function, median, k^{th} moment, moment generating function, mean residual life function, mean waiting time, and Rényi entropy, are all derived in this section.

A. Quantile function

The Quantile function $\Big(Q(p)\Big)$ of the EAPIEx distribution is the inverse of eq.(6) for $p \in (0,1)$.

$$Q(p) = -\frac{\beta}{\log\left\{\frac{\log\left(1+p^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}} \tag{16}$$

Proof:

Let $G(y) = p$, for $p \in (0,1)$ and then solve for y;

$$p = \left(\frac{\alpha^{e^{-\beta y^{-1}}}-1}{\alpha-1}\right)^{\mu}$$

$$\alpha^{e^{-\beta y^{-1}}} - 1 = p^{\frac{1}{\mu}}(\alpha-1)$$

$$\alpha^{e^{-\beta y^{-1}}} = 1 + p^{\frac{1}{\mu}}(\alpha-1)$$

$$e^{-\beta y^{-1}}\log\alpha = \log\left(1+p^{\frac{1}{\mu}}(\alpha-1)\right)$$

$$\frac{-\beta}{y} = \log\left\{\frac{\log(1+p^{\frac{1}{\mu}}(\alpha-1))}{\log\alpha}\right\}$$

$$y = -\frac{\beta}{\log\left\{\frac{\log\left(1+p^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}}$$

The random variables of the EAPIEx distribution can be generated/calculated by using eq.(16);

$$y_p = -\frac{\beta}{\log\left\{\frac{\log\left(1+p^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}} \quad ;\text{at } \ p \in (0,1) \tag{17}$$

a. Lower (1^{st}) quantile

By using $p = 1/4$ in eq.(16), the Lower (1^{st}) quantile of the EAPIEx is given as;

$$Q(1/4) = -\frac{\beta}{\log\left\{\frac{\log\left(1+(1/4)^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}} \tag{18}$$

b. Median (2^{nd}) quantile

By making use of $p = 1/2$ in eq.(16) the Median (2^{nd}) quantile of the EAPIEx is given as;

$$Q(1/2) = -\frac{\beta}{\log\left\{\frac{\log\left(1+(1/2)^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}} \tag{19}$$

c. Upper (3^{rd}) quantile

The Upper (3^{rd}) quantile of the EAPIEx is derived by making use of $p = 3/4$ in eq.(16);

$$Q(3/4) = -\frac{\beta}{\log\left\{\frac{\log\left(1+(3/4)^{\frac{1}{\mu}}(\alpha-1)\right)}{\log\alpha}\right\}} \tag{20}$$

Its clear in Table 1 and Figure 4 that the quantile values are directly proportional to the probability values.

Table 1: Quantile Values for EAPIEx(α,μ,β) distribution.

p	(2.0, 1.1, 0.9)	(2.1, 1.2, 0.8)	(2.2, 1.3, 0.7)	(2.3, 1.4, 0.6)
0.1	0.5041	0.4998	0.4842	0.4568
0.2	0.7484	0.7432	0.7209	0.6804
0.3	1.0276	1.0208	0.9903	0.9347
0.4	1.3795	1.3703	1.3291	1.2540
0.5	**1.8570**	**1.8440**	**1.7879**	**1.6862**
0.6	2.5597	2.5409	2.4626	2.3214
0.7	3.7175	3.6886	3.5732	3.3669
0.8	6.0172	5.9675	5.7781	5.4421
0.9	12.8903	12.7776	12.3663	11.6420

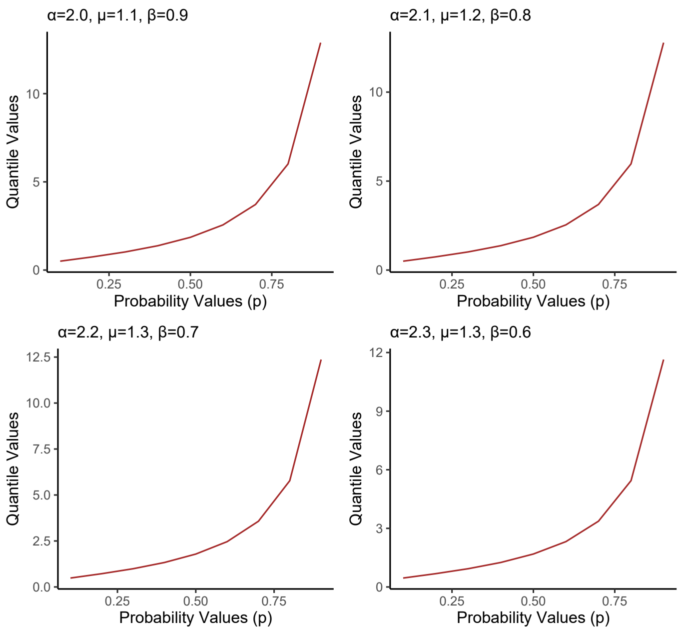


Figure 4: The Quantile Values plots of the EAPIEx distribution

B. k^{th} moments

The distribution's location, spread, asymmetry, and tail behavior can all be numerically measured using the moments of the distribution.

The k^{th} moment of the EAPIEx distribution is defined as;

$$E(Y^k) = \omega_k' = \int_0^\infty y^k g(y)dy \tag{23}$$

By utilizing $g(y)$ from eq.(7) in eq.(23), the k^{th} moment is given as;

$$E(Y^k) = \omega_k' = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{k-2} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy \tag{24}$$

Proof:

$$\omega_k' = \int_0^\infty y^k \Phi y^{-2} \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

$$= \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{k-2} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

a. Mean

The mean is the 1^{st} moment of a distribution. The mean of the EAPIEx can be derived by putting $k=1$ in eq.(24).

$$E(Y) = \omega' = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy \tag{25}$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

b. Variance

The variance of a distribution is an indicator of the spread or dispersion of values within it, and the variance of EAPIEx distribution is defined as;

$$\begin{aligned} Var(Y) &= E(Y^2) - (E(Y))^2 \\ &= \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy - \left\{ \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy \right\}^2 \end{aligned} \tag{26}$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

c. Coefficient of Variation

The coefficient of variation (CV) is a statistical measure used to estimate the relative variability of a data collection or distribution. The CV of the EAPIEx is defined as;

$$CV = \frac{Var(Y)}{E(Y)} \tag{27}$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

$$Var(Y) = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy - \left\{ \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy \right\}^2$$

and

$$E(Y) = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

C. Moment generating function

The moment generating function (mgf) is a mathematical function that offers a means of describing the distribution of a random variable.

$$M_Y(t) = E\left(e^{tY}\right) \tag{28}$$

Utilizing the identity; $e^{tY} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Y^k$

$$\begin{aligned} M_Y(t) &= E\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} Y^k\right) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(Y^k) \end{aligned} \tag{29}$$

Hence, utilizing $E(Y^k)$ from eq.(24) in eq.(29), the mgf is given as;

$$\begin{aligned} M_Y(t) = \Phi \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1} \frac{t^k}{k!}(-1)^j \binom{\mu-1}{j} \frac{(\log\alpha)^{i+1}}{i!} \\ \int_0^{\infty} y^{k-2}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy \end{aligned} \tag{30}$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

D. Rényi Entropy

The Rényi entropy offers a means to measure the uncertainty in a probability distribution and is a generalization of the Shannon entropy.

The Rényi entropy of the EAPIEx is defined as;

$$R_{\Upsilon}(Y) = \frac{1}{1-\Upsilon}\log\left(\int_0^{\infty} g^{\Upsilon}(y)dy\right) \quad \Upsilon \neq 1, (>0) \tag{35}$$

For the random variable Y, Rényi entropy of the EAPIEx distribution is given as;

$$\begin{aligned} R_{\Upsilon}(Y) = \frac{1}{1-\Upsilon}\log\Bigg\{\Phi^{\Upsilon}\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}\int_0^{\infty}\Bigg[(-1)^j\binom{\mu-1}{j}y^{-2} \\ \frac{(\log\alpha)^{i+1}}{i!}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}\Bigg]^{\Upsilon} dy\Bigg\} \end{aligned} \tag{36}$$

Proof:
Making use of $g(y)$ from eq.(7) in eq.(35);

$$\begin{aligned} R_{\Upsilon}(Y) &= \frac{1}{1-\Upsilon}\log\Bigg\{\int_0^{\infty}\Bigg[\Phi y^{-2}\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j} \\ &\quad \frac{(\log\alpha)^{i+1}}{i!}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}\Bigg]^{\Upsilon} dy\Bigg\} \\ &= \frac{1}{1-\Upsilon}\log\Bigg\{\Phi^{\Upsilon}\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}\int_0^{\infty}\Bigg[(-1)^j\binom{\mu-1}{j} \\ &\quad y^{-2}\frac{(\log\alpha)^{i+1}}{i!}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}\Bigg]^{\Upsilon} dy\Bigg\} \end{aligned}$$

where; $\Phi = \mu\beta(\alpha-1)^{-\mu}$

Through the help of numerical integration using R-software, the values of the Rényi entropy of the EAPIEx(α,μ,β) distribution was simulated and presented in Table 2 and its clear in Table 2 & Figure 5 that the order (γ) is inversely proportional to the Rényi entropy value.

Table 2: The values for Rényi entropy of the EAPIEx(α,μ,β) distribution.

$R_{\Upsilon}(Y)$	(2.5, 2.0, 3.0)	(2.2, 1.7, 2.7)	(1.9, 1.4, 2.4)	(1.6, 1.1, 2.1)
$R_{0.5}(Y)$	14.0448	13.7237	13.3495	12.8996
$R_{0.9}(Y)$	4.5708	4.24570	3.86640	3.4113
$R_{1.3}(Y)$	3.9786	3.6523	3.2716	2.8152
$R_{1.7}(Y)$	3.7117	3.3847	3.0031	2.5459
$R_{2.1}(Y)$	3.5545	3.2270	2.8448	2.3870
$R_{2.5}(Y)$	3.4491	3.1214	2.7388	2.2805
$R_{2.9}(Y)$	3.3729	3.0449	2.6620	2.2034
$R_{3.3}(Y)$	3.3148	2.9866	2.6034	2.1447

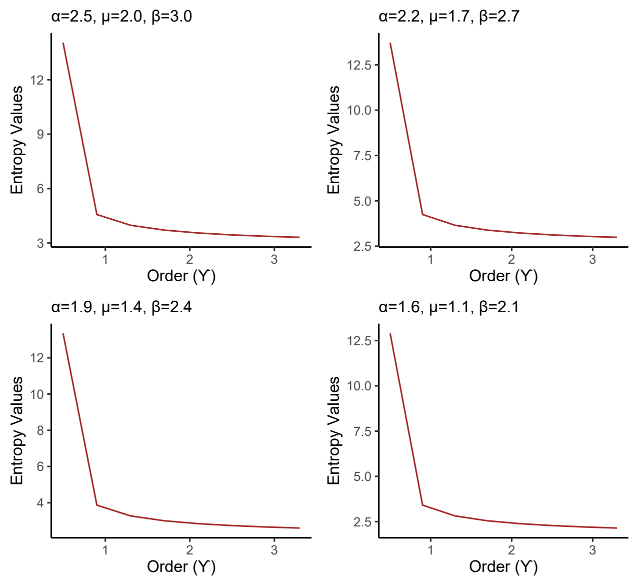


Figure 5: The entropy values plots of the EAPIEx distribution

E. Mean Residual Life function

A key idea in survival research and reliability theory is the Mean Residual Life (MRL) function. The fact that a system or entity has made it this far gives crucial information about how long it will live.

The MRL function is denoted as $\mu(t)$;

$$\begin{aligned}\mu(t) &= \frac{1}{P(Y>t)}\int_t^{\infty} P(Y>y)dy \\ &= \frac{1}{S(t)}\left(E(t)-\int_0^t yg(y)dy\right)-t \;\; ;t\geq 0\end{aligned} \tag{31}$$

$$\mu(t)=\frac{\mu\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\left\{\int_0^{\infty}t^{-1}\left(e^{-\beta t^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-1-j}dt-\int_0^t y^{-1}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}dy\right\}-t\left\{(\alpha-1)^{\mu}-\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}\right\}}{(\alpha-1)^{\mu}-\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}} \tag{32}$$

Proof:
Making use of;
$S(t)=1-(\alpha-1)^{-\mu}\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}$,
$E(t)=\Phi\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\int_0^{\infty}t^{-1}\left(e^{-\beta t^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-1-j}dt$

$\int_0^t yg(y)dy=\Phi\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\int_0^t y^{-1}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}dy$
and
$\Phi=\mu\beta(\alpha-1)^{-\mu}$

From eq.(31);

$$\begin{aligned}\mu(t) &= \frac{1}{1-(\alpha-1)^{-\mu}\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}}\left\{\Phi\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\int_0^{\infty}t^{-1}\left(e^{-\beta t^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-1-j}dt\right. \\ &\quad \left.-\Phi\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\int_0^t y^{-1}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}dy\right\}-t \\ &= \frac{(\alpha-1)^{\mu}\Phi\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}}{(\alpha-1)^{\mu}-\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}}\left\{\int_0^{\infty}t^{-1}\left(e^{-\beta t^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-1-j}dt-\int_0^t y^{-1}\left(e^{-\beta y^{-1}}\right)^{i+1}\right. \\ &\quad \left.\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}dy\right\}-t\end{aligned}$$

$$=\frac{\mu\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\mu-1}(-1)^j\binom{\mu-1}{j}\frac{(\log\alpha)^{i+1}}{i!}\left\{\int_0^{\infty}t^{-1}\left(e^{-\beta t^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-1-j}dt-\int_0^t y^{-1}\left(e^{-\beta y^{-1}}\right)^{i+1}\left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j}dy\right\}-t\left\{(\alpha-1)^{\mu}-\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}\right\}}{(\alpha-1)^{\mu}-\sum_{v=0}^{\mu}(-1)^v\binom{\mu}{v}\left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}}$$

F. Mean Waiting Time (MWT)

The mean waiting time measures how long it takes on average for consumers or other organizations to receive service after joining a queue.

The MWT is denoted as $\bar{\mu}(t)$;

$$\bar{\mu}(t) = t - \frac{1}{G(t)} \int_0^\infty y g(y) dy \quad ; t \geq 0 \tag{33}$$

$$\bar{\mu}(t) = \frac{t \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v} - \mu\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy}{\sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}} \tag{34}$$

Proof:

Making use of;

$G(t) = (\alpha-1)^{-\mu} \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}$

$\int_0^\infty y g(y) dy = \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$

and

$\Phi = \mu\beta(\alpha-1)^{-\mu}$

From eq.(33);

$$\bar{\mu}(t) = t - \frac{\Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!}}{(\alpha-1)^{-\mu} \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

$$= t - \frac{(\alpha-1)^{\mu}}{\sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}} \Phi \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

$$= t - \frac{\mu\beta}{\sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}} \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy$$

$$= \frac{t \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v} - \mu\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\mu-1} (-1)^j \binom{\mu-1}{j} \frac{(\log \alpha)^{i+1}}{i!} \int_0^\infty y^{-1} \left(e^{-\beta y^{-1}}\right)^{i+1} \left(\alpha^{e^{-\beta y^{-1}}}\right)^{\mu-1-j} dy}{\sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} \left(\alpha^{e^{-\beta t^{-1}}}\right)^{\mu-v}}$$

IV. Estimation of Parameters

For random variables $Y_1, Y_2, Y_3, \cdots, Y_n$, let the random sample size be 'n' for Y ∼EAPIEx(ϕ), where $\phi \in (\alpha, \mu, \beta)$, the joint probability density function is given as;

$$L(y_i; \phi) = \prod_{i=0}^{n} g(y_i)$$

$$L(y_i; \phi) = \prod_{i=0}^{n} \left[\frac{\mu\beta \log \alpha}{y_i^2 (\alpha-1)^{\mu}} e^{-\beta y_i^{-1}} \alpha^{e^{-\beta y_i^{-1}}} \left(\alpha^{e^{-\beta y_i^{-1}}} - 1\right)^{\mu-1} \right] \tag{37}$$

The log-likelihood of eq.(37) becomes;

$$l(\phi) = \ln\left[\frac{\mu\beta\log\alpha}{(\alpha-1)^{\mu}}\right]^n - 2\sum_{i=0}^{n}\ln(y_i) - \beta\sum_{i=0}^{n} y_i^{-1} + \ln(\alpha)\times \sum_{i=0}^{n} e^{-\beta y_i^{-1}} + (\mu-1)\ln\left(\alpha^{\sum_{i=0}^{n} e^{-\beta y_i^{-1}}} - 1\right)$$

$$l(\phi) = n\ln(\mu) + n\ln(\beta) + n\ln(\log(\alpha)) - n\mu\ln(\alpha-1) - 2\sum_{i=0}^{n}\ln(y_i) - \beta\sum_{i=0}^{n} y_i^{-1} + \ln(\alpha)\sum_{i=0}^{n} e^{-\beta y_i^{-1}} + (\mu-1)\ln\left(\alpha^{\sum_{i=0}^{n} e^{-\beta y_i^{-1}}} - 1\right) \quad (38)$$

Differentiating eq.(38) partially wrt for each parameter (α, μ, β) and equate to zero;

$$\text{Let;}\quad \Theta = (\mu-1)\ln\left(\alpha^{\sum_{i=0}^{n} e^{-\beta y_i^{-1}}} - 1\right) \quad (39)$$

From eq.(39);

$$\Theta'_\alpha = \frac{\partial\Theta}{\partial\alpha},\quad \Theta'_\mu = \frac{\partial\Theta}{\partial\mu},\quad \Theta'_\beta = \frac{\partial\Theta}{\partial\beta}$$

Hence;

$$\frac{\partial l(\phi)}{\partial\alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n\mu}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n} e^{-\beta y_i^{-1}} + \Theta'_\alpha = 0 \quad (40)$$

$$\frac{\partial l(\phi)}{\partial\mu} = \frac{n}{\mu} - n\ln(\alpha-1) + \Theta'_\mu = 0 \quad (41)$$

$$\frac{\partial l(\phi)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} y_i^{-1} - \ln(\alpha)\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right) + \Theta'_\beta = 0 \quad (42)$$

eqs.(40)-(42) are non linear, hence we need a numerical optimization method to find their solutions.

Here we used a numerical optimization method called the BFGS algorithm, and the observed information matrix is given as;

$$J^{-1}(\psi) = \begin{Bmatrix} \frac{\partial^2 l(\phi)}{\partial\alpha^2} & \frac{\partial^2 l(\phi)}{\partial\alpha\partial\mu} & \frac{\partial^2 l(\phi)}{\partial\alpha\partial\beta} \\ & \frac{\partial^2 l(\phi)}{\partial\mu^2} & \frac{\partial^2 l(\phi)}{\partial\mu\partial\beta} \\ & & \frac{\partial^2 l(\phi)}{\partial\beta^2} \end{Bmatrix}$$

where; $\psi = (\alpha, \mu, \beta)'$

The expressions for the information matrix are available in the Appendix

V. Monte Carlo Simulation

To assess the average bias, the root mean square error, and the average of the parameter estimates, a simulation exercise was carried out. The simulation used several sample sizes and sets of initial parameter values.

The simulation was performed under 2000 replications for each sample size (n)= $50, 100, 150, \ldots, 500$ and eq.(17) were used to generate the random samples for the simulation study.

The average bias and root mean square error are calculated using eqs.(43) & (44), respectively;

$$\text{AB} = \frac{1}{P}\sum_{i=1}^{P}(\widehat{d}_i - d) \quad (43)$$

and

$$\text{RMSE} = \sqrt{\frac{1}{P}\sum_{i=1}^{P}(\widehat{d}_i - d)^2} \quad (44)$$

where $d \in (\alpha, \mu, \beta)$ and P the number of replications

Table 3: The estimates, ABs, and RMSEs for EAPIEx($\alpha = 1.30, \mu = 0.30, \beta = 0.02$)

n	Estimates			ABs		
	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$
50	2.0507	0.4171	0.5995	0.7507	0.1171	0.5795
100	1.6402	0.4029	0.4126	0.3402	0.1029	0.3926
150	1.5168	0.4084	0.2766	0.2168	0.10840	0.2566
200	1.4357	0.3913	0.2163	0.1357	0.0913	0.1963
250	1.4149	0.3792	0.1628	0.1149	0.0792	0.1428
300	1.3917	0.3811	0.1429	0.0917	0.0811	0.1229
350	1.3770	0.3697	0.1201	0.0770	0.0697	0.1001
400	1.3755	0.3703	0.1096	0.0755	0.0703	0.0896
450	1.3597	0.3648	0.1037	0.0597	0.0648	0.0837
500	1.3494	0.3554	0.1062	0.0494	0.0554	0.0862

n	RMSEs		
	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$
50	3.9156	0.3376	0.7469
100	1.3228	0.2950	0.5936
150	0.8211	0.3553	0.4534
200	0.4721	0.3061	0.3699
250	0.4875	0.2749	0.2776
300	0.3314	0.2942	0.2352
350	0.3155	0.2691	0.1714
400	0.2961	0.2728	0.1378
450	0.2525	0.2648	0.1098
500	0.2138	0.2327	0.1165

Table 4: The estimates, ABs, and RMSEs for EAPIEx($\alpha = 1.20, \mu = 0.20, \beta = 0.01$)

n	Estimates			ABs		
	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$
50	1.3498	0.2073	0.0951	0.1498	0.0073	0.0851
100	1.2224	0.2044	0.0881	0.0224	0.0044	0.0781
150	1.2045	0.2043	0.0776	0.0045	0.0043	0.0676
200	1.2039	0.2038	0.0708	0.0039	0.0038	0.0608
250	1.2044	0.2035	0.0618	0.0044	0.0035	0.0518
300	1.2053	0.2040	0.0569	0.0053	0.0040	0.0469
350	1.2044	0.2043	0.0520	0.0044	0.0043	0.0420
400	1.2044	0.2031	0.0460	0.0044	0.0031	0.0360
450	1.2034	0.2033	0.0455	0.0034	0.0033	0.0355
500	1.2030	0.2029	0.0403	0.0030	0.0029	0.0303

n	RMSEs		
	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$
50	1.0566	0.0405	0.0925
100	0.2348	0.0372	0.0883
150	0.0798	0.0311	0.0819
200	0.0718	0.0205	0.0775
250	0.0755	0.0192	0.0715
300	0.0756	0.0232	0.0680
350	0.0664	0.0294	0.0644
400	0.0672	0.0142	0.0595
450	0.0568	0.0215	0.0591
500	0.0546	0.0264	0.0547

Remarks on Simulation Results

1. As seen in Table 3 and Table 4, as the sample size (n) increases, the average parameter estimates converge to the initial parameter value.
2. As the sample size goes higher, the Average Bias (AB) decreases.
3. The sample size has an inverse relationship with the Root Mean Square Error (RMSE).
4. The sensitive of α depends on the sample size (n).
5. The estimators work well and offer low values for the RMSE, AB, and average parameter values as they approach the initial parameter values.

VI. Data Application

By utilizing UK COVID-19 Mortality Rate Data, this section of the paper evaluates the applicability and flexibility of the EAPIEx distribution.
The EAPIEx distribution is compared with the following distributions:

• Alpha Power Exponentiated Inverse Rayleigh (APEIR) distribution by [19]
The cdf and pdf are defined respectively as;

$$G(y) = \frac{\alpha^{e^{-\frac{\beta\theta}{y^2}}} - 1}{\alpha - 1}$$
$$g(y) = \frac{2\beta\theta\log(\alpha)}{y^3(\alpha-1)} e^{-\frac{\beta\theta}{y^2}} \alpha^{e^{-\frac{\beta\theta}{y^2}}}$$

• Exponentiated Transmuted Inverse Exponential (ETIEx) distribution by [20]
The cdf and pdf are defined respectively as;

$$G(y) = \left(e^{-\frac{a}{y}}\left(1+\lambda-\lambda e^{-\frac{a}{y}}\right)\right)^b$$
$$g(y) = \frac{ab}{y^2} e^{-\frac{ab}{y}}\left(1+\lambda-2\lambda e^{-\frac{a}{y}}\right)\left(1+\lambda-\lambda e^{-\frac{a}{y}}\right)^{b-1}$$

where; $-1 \le \lambda \le 1$
• Exponentiated Inverse Exponential (EIEx) distribution
The cdf and pdf are defined respectively as;

$$G(y) = e^{-\frac{\mu\beta}{y}}$$
$$g(y) = \beta\mu y^{-2} e^{-\frac{\mu\beta}{y}}$$

• Inverse Exponential (IEx) distribution
The cdf and pdf are defined respectively as;

$$G(y) = e^{-\frac{\beta}{y}}$$
$$g(y) = \beta y^{-2} e^{-\frac{\beta}{y}}$$

A. Data Set: United Kingdom COVID-19 Mortality Rates Data

Table 5, consist of COVID-19 mortality rates data set in the United Kingdom used by [21].

Table 5: United Kingdom COVID-19 Mortality Rates Data.

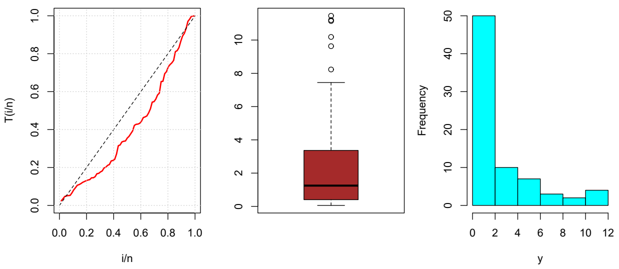
0.0587	0.0863	0.1165	0.1247	0.1277	0.1303
0.1652	0.2079	0.2395	0.2845	0.2992	0.3188
0.3317	0.3446	0.3553	0.3622	0.3926	0.3926
0.4633	0.4690	0.4954	0.5139	0.5696	0.5837
0.6197	0.6365	0.7096	0.7444	0.8590	1.0438
1.0602	1.1305	1.1468	1.1533	1.2260	1.2707
1.4149	1.5709	1.6017	1.6083	1.6324	1.6998
1.8164	1.8392	1.8721	2.1360	2.3987	2.4153
2.5225	2.7087	2.7946	3.3609	3.3715	3.7840
4.1969	4.3451	4.4627	4.6477	5.3664	5.4500
5.7522	6.4241	7.0657	8.2307	9.6315	10.1870
11.1429	11.2019	11.4584	0.2751	0.4110	0.7193
1.3423	1.9844	3.9042	7.4456		

The descriptive statistics for UK COVID-19 mortality rates is presented in Table 6. The data is right-skewed, is unimodal, and have a value for kurtosis < 3 (Platykurtic).

Table 6: Descriptive Statistics of UK COVID-19 Mortality Rates Data.

N	Max.	Min.	Mean	Median	Mode	SD	Skew.	Kurt.
76	11.46	0.06	2.44	1.25	0.39	2.94	1.74	2.32

The failure rate of UK COVID-19 mortality rate data has bathtub shape, as illustrated in Figure 6, because the TTT-transform plot is initially convex below the 45° line and subsequently concave above the 45° line, with few outliers in the boxplot.



(a) TTT-transform (b) Boxplot (c)Histogram

Figure 6: The Boxplot, the Histogram and the TTT-transform plot of the Data Set.

The asymptotic variance-covariance matrix of the EAPIEx distribution parameters for the UK COVID-19 Mortality Rate Data is given as;

$$J^{-1}(\psi) = \begin{Bmatrix} 51.5770 & -0.3867 & -0.1271 \\ -0.3867 & 0.0865 & -0.1198 \\ -0.1271 & -0.1198 & 0.1825 \end{Bmatrix}$$

where $\psi = (\alpha, \mu, \beta)'$

The maximum likelihood with a standard in brackets of the EAPIEx distribution and other well-known distributions are provided in Table 7, majority of the fitted distributions have standard error less than half the parameter value.

Table 7: The MLEs and the Standard Error (in parentheses).

Distributions			
EAPIEx$(\hat{\alpha}, \hat{\mu}, \hat{\beta})$	14.4870(11.5240)	0.5180(0.2722)	0.5943(0.3848)
APEIR$(\hat{\alpha}, \hat{\theta}, \hat{\beta})$	5.3559(1.3252)	5.1246(2.4636)	0.0133(0.0064)
ETIEx$(\hat{\lambda}, \hat{b}, \hat{a})$	−0.9676(0.0456)	0.1368(0.0628)	2.6458(1.3610)
EIEx$(\hat{\mu}, \hat{\beta})$	3.0760(36.4912)	0.1676(1.9880)	
IEx$(\hat{\beta})$	0.5155(0.0591)		

With regards to UK COVID-19 Mortality Rate Data, the EAPIEx distribution in Table 8 has the largest log-likelihood, $p - value$, and the smallest Kolmogorov-Smirnov test $(K - S)$, its implies that the EAPIEx provide a better fit than the other distributions used in this study.

Table 8: The $l(\phi)$, the Goodness-of-fit and the $p - values$ results.

Distributions	$l(\phi)$	$K - S$	$p - value$	$Rank$
EAPIEx	**−142.4631**	**0.0790**	**0.7294**	1^{st}
APEIR	−220.1216	0.6892	<5%	5^{th}
ETIEx	−143.4585	0.10293	0.3965	2^{nd}
EIEx	−149.6024	0.1892	<5%	3^{rd}
IEx	−149.6024	0.1892	<5%	3^{rd}

Table 9 displays the data information criterion, and it is obvious that the EAPIEx distribution has the lowest AIC, AICc, BIC, and HQIC when compared to the other distributions. In model selection, it is stated that for a model to be selected over other models, it must have the lowest information criterion values. In this case, the EAPIEx distribution is the optimal model for simulating the UK COVID-19 mortality rate in terms of the model selection principle.

Table 9: The Information Criteria results.

Distributions	AIC	$AICc$	BIC	$HQIC$
EAPIEx	**290.9263**	**291.2596**	**297.9185**	**293.7207**
APEIR	446.2433	446.5766	453.2355	449.0377
ETIEx	292.9171	293.2504	299.9093	295.7115
EIEx	303.2048	303.3692	307.8663	305.0678
IEx	301.2048	301.2589	303.5355	302.1363

According to Figure 7, the EAPIEx distribution fits the COVID-19 mortality rate data in the United Kingdom better than other well-known distributions.

VII. Conclusion

The major goal of this research was to create a new three-parameter distribution termed "the Exponentiated Alpha Power Inverted Exponential distribution, abbreviated as EAPIEx distribution", which was used to model UK COVID-19 Mortality Rate Data. Various statistical properties are derived and studied such as Quantile function, Lower quantile, Median, Upper quantile, Survival function, Hazard function, k^{th} Moments, Moment generating function, and Entropy. The parameters of the new distribution are estimated using the maximum likelihood method, and the performance of the estimation method is investigated using Monte Carlo simulation. The average parameter estimate, average bias, and root mean square error are calculated, and it is evident that the sample size (n) is inversely related to the estimate (that is the estimate converges to the initial parameter value as the sample size increases).

Regarding the main purpose of this paper, we applied the distribution to UK COVID-19 Mortality Rate Data. It demonstrates that the EAPIEx distribution outperforms the other distributions employed in this investigation.

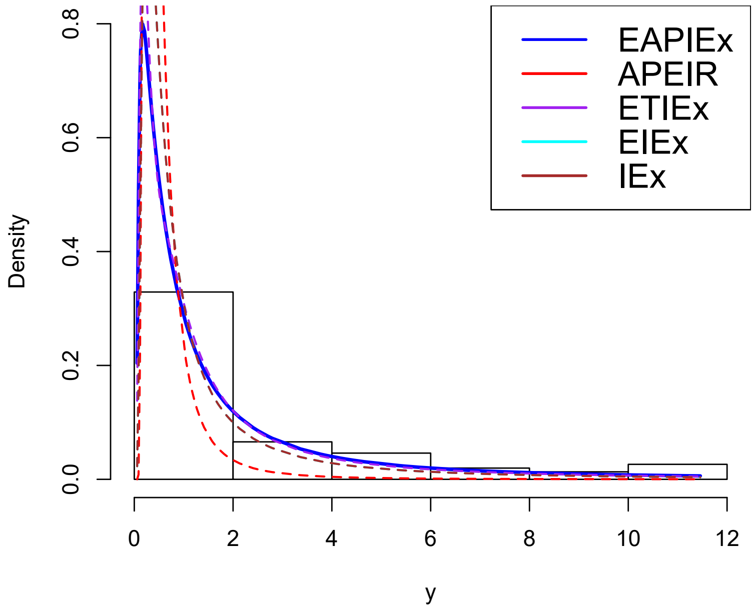


Figure 7: **The Fitted densities plot of UK COVID-19 Mortality Rates.**

Conflicts of Interest

The authors have no conflicts of interest to disclose.

Data availability

The real data set used for the application of this proposed distribution is within the manuscript (Table 5).

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Appendix

Expressions of the Information matrix are derived below;

From eqs. (40) to (42)

Note;

$$\frac{\partial^2\Theta}{\partial\alpha^2} = \Theta''_{\alpha},$$

$$\frac{\partial^2\Theta}{\partial\alpha\partial\mu} = \Theta''_{\alpha\mu},$$

$$\frac{\partial^2\Theta}{\partial\alpha\partial\beta} = \Theta''_{\alpha\beta},$$

$$\frac{\partial^2\Theta}{\partial\mu^2} = \Theta''_{\mu},$$

$$\frac{\partial^2\Theta}{\partial\mu\partial\beta} = \Theta''_{\mu\beta},$$

$$\frac{\partial^2\Theta}{\partial\beta^2} = \Theta''_{\beta}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\alpha^2} &= \frac{\partial}{\partial\alpha}\left(\frac{\partial l(\phi)}{\partial\alpha}\right)\\ &= \frac{\partial}{\partial\alpha}\left(\frac{n}{\alpha\log(\alpha)} - \frac{n\mu}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n} e^{-\beta y_i^{-1}} + \Theta'_{\alpha}\right)\\ &= \frac{n\mu}{(\alpha-1)^2} - \frac{n(1+\log\alpha)}{(\alpha\log\alpha)^2} + \frac{1}{\alpha^2}\sum_{i=1}^{n} e^{-\beta y_i^{-1}} + \Theta''_{\alpha}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\alpha\partial\mu} &= \frac{\partial}{\partial\alpha}\left(\frac{\partial l(\phi)}{\partial\mu}\right)\\ &= \frac{\partial}{\partial\alpha}\left(\frac{n}{\mu} - n\ln(\alpha-1) + \Theta'_{\mu}\right)\\ &= -\frac{n}{\alpha-1} + \Theta''_{\alpha\mu}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\alpha\partial\beta} &= \frac{\partial}{\partial\alpha}\left(\frac{\partial l(\phi)}{\partial\beta}\right)\\ &= \frac{\partial}{\partial\alpha}\left(\frac{n}{\beta} - \sum_{i=1}^{n} y_i^{-1} - \ln(\alpha)\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right) + \Theta'_{\beta}\right)\\ &= -\frac{1}{\alpha}\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right) + \Theta''_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\mu^2} &= \frac{\partial}{\partial\mu}\left(\frac{\partial l(\phi)}{\partial\mu}\right)\\ &= \frac{\partial}{\partial\mu}\left(\frac{n}{\mu} - n\ln(\alpha-1) + \Theta'_{\mu}\right)\\ &= -\frac{n}{\mu^2} + \Theta''_{\mu}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\mu\partial\beta} &= \frac{\partial}{\partial\mu}\left(\frac{\partial l(\phi)}{\partial\beta}\right)\\ &= \frac{\partial}{\partial\mu}\left(\frac{n}{\beta} - \sum_{i=1}^{n} y_i^{-1} - \ln(\alpha)\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right) + \Theta'_{\beta}\right)\\ &= \Theta''_{\mu\beta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l(\phi)}{\partial\beta^2} &= \frac{\partial}{\partial\beta}\left(\frac{\partial l(\phi)}{\partial\beta}\right)\\ &= \frac{\partial}{\partial\beta}\left(\frac{n}{\beta} - \sum_{i=1}^{n} y_i^{-1} - \ln(\alpha)\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right) + \Theta'_{\beta}\right)\\ &= -\frac{n}{\beta^2} + \ln(\alpha)\sum_{i=1}^{n}\left(y_i^{-1}e^{-\beta y_i^{-1}}\right)^2 + \Theta''_{\beta}\end{aligned}$$

