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On The Exponentiated Generalized Exponentiated Exponential Distribution with Properties and Application

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Abstract:

This article proposes the exponentiated generalized exponentiated exponential distribution (written as EGEEx), which has four parameters (i.e., three shape parameters and one scale parameter). Some well-known sub-models of the EGEEx distribution are derived using the EGEEx cumulative distribution function. A few essential statistical properties of the proposed EGEEx distribution are derived. The method of maximum likelihood estimation (MLE) is used to estimate the parameters of the EGEEx distribution. The performance of the MLE estimates is assessed by employing a Monte Carlo simulation study. The results of the simulation study show that an increase in sample size leads to a decrease in average estimates (i.e., average estimates converge to the true parameter values), and the average bias, mean square error, and root mean square error also decrease as n increases. Real data is used to fit the EGEEx distribution and its sub-models and compare their performances. A conclusion can be drawn from the fitted data set that the proposed EGEEx distribution performs better than its sub-models.

Keywords - Exponential distribution, Exponentiated, Generalized, Maximum Likelihood Estimation

I. INTRODUCTION

In statistics, the basic concept that is commonly used practically and/or theoretically is a probability distribution. In most areas of study, researchers have relied on data analysis to communicate, but some of these data may show behaviour such as monotonic hazard rates, non-monotonic failure rates, skewness, kurtosis, etc. With such data characteristics, a large number of distributions and families of distributions, especially the classical distributions such as the exponential distribution, Rayleigh distribution, and the family of distributions such as the Alpha Power family of distributions, the Gull Alpha Power family of distributions, etc., are incapable of handling more than one of the data behavours due to the single shape parameter of their cumulative distribution function (CDF).

Mathematicians and statisticians, on the other hand, have introduced methods on the fundamental or classical distributions and on some weaker families of distributions by introducing new distributions that are flexible and tangible enough to accommodate two or more data characteristics simultaneously; this method was accomplished by adding extra shape parameter(s) to the classical or fundamental distributions.
The method of adding extra shape parameters to distributions has been accomplished in several ways, such as the combination of two or more distributions method, the quantile method, variable transformation methods, and the exponentiated method, which involves raising the cumulative distribution function (CDF) to a parameter introduced by [1].

With the aid of the above methods, new distributions and families of distributions have been intro-

duced, the exponentiated expontiatal distribution introduced by [2], the exponentiated Rayleigh distribution using Bayesian analysis introduced by [3], the exponentiated generalized class of distributions introduced by [4], Statistical properties of the exponentiated generalized inverted exponential (EGIE) distribution introduced by [5], Slashed exponentiated rayleigh distribution introduced by [6], Assessing the flexibility of the exponentiated generalized exponential (EGE) distribution introduced by [7], The exponentiated generalized extended exponential ($\varepsilon_g \varepsilon\varepsilon$) distribution introduced by [8], the exponentiated Rayleigh distribution based on generalized Type-II hybrid censored data introduced by [9], exponentiated Rayleigh distribution using MCMC approach introduced by [10], the generalized Marshall-Olkin Kumaraswamy-G family introduced by [11], the Kumaraswamy alpha power-G family introduced by [12], An extended Kumaraswamy-Gull Alpha Power Exponential (K-GAPE) distribution introduced by [13], the exponentiated generalized gull alpha power exponential (EGGAPE) distribution introduced by [14], the Kumaraswamy-Gull Alpha Power Rayleigh (Kw-GAPR) distribution introduced by [15].
This paper introduces the EGEEx distribution, which has four parameters.

In 2013, [4] defined the CDF, $F(x)$ and PDF, $f(x)$ of the Exponentiated Generalized (EG) family of distributions as:

$$F(x) = [1 - \{1 - M(x)\}^a]^\alpha \tag{1}$$

$$f(x) = \alpha a m(x)\{1 - M(x)\}^{a-1}[1 - \{1 - M(x)\}^a]^{\alpha-1} \tag{2}$$

for $a, \alpha > 0$ are the shape parameters.
While the CDF and PDF of the Exponentiated exponential (EEx) distribution defined by [2] is given as

$$M(x) = (1 - e^{-\beta x})^b \tag{3}$$

$$m(x) = \beta b e^{-\beta x}(1 - e^{-\beta x})^{b-1} \tag{4}$$

for $b > 0$ is the shape parameter and $\beta > 0$ being the scale parameter.

The CDF and PDF of the exponentiated exponential (EEx) distribution are used as baseline distributions to develop the EGEEx distribution, which has four parameters. The main objective of creating the EGEEx distribution is to provide more flexibility by handling two or more of the characteristics that data sets usually possess, such as unimodality, bimodality, skewness, kurtosis, monotonic, and non-monotonic behaviors that are normally encountered in the real world.

The remaining part of this paper is arranged as follows:
Section “Developing the distribution" presents the proposed EGEEx distribution, and its sub-models, Section “Statistical properties" presents some statistical properties of the EGEEx distribution, Section “Parameters Estimation" presents the maximum likelihood estimation (MLE) and the simulation study results of the EGEEx parameters. The proposed EGEEx distribution is then used with its competing models to analyze real data set in Section “Application of data", and Section “conclusion" presents the concluding remarks about the EGEEx.

II. DEVELOPING THE DISTRIBUTION

A. The Exponentiated Generalized Exponentiated Exponential (EGEEx) Distribution

The Exponentiated Generalized Exponentiated Exponential (EGEEx) distribution CDF is obtained by substituting Eq. (3) into Eq. (1), and Eqs. (3) and (4) into Eq. (2) to obtain the EGEEx PDF.
Eqs. (5) and (6) show the CDF and PDF of the EGEEx distribution.

$$F_{EGEEx}(x) = \left[1 - \left\{1 - (1 - e^{-\beta x})^b\right\}^a\right]^\alpha \tag{5}$$

$$f_{EGEEx}(x) = \alpha\beta a b e^{-\beta x}\left(1 - e^{-\beta x}\right)^{b-1} \left\{1 - (1 - e^{-\beta x})^b\right\}^{a-1}\left[1 - \left\{1 - (1 - e^{-\beta x})^b\right\}^a\right]^{\alpha-1} \tag{6}$$

for $x > 0$, $\alpha, a, b > 0$ are the shape parameters, and $\beta > 0$ being the scale parameter.
The survival function (S(x)) and hazard rate function (h(x)) of the EGEEx are given as

$$S(x) = 1 - F_{EGEEx}(x)$$

and

$$h(x) = \frac{f_{EGEEx}(x)}{S(x)}$$

The density shapes of the EGEEx distribution are shown in Fig. 1. These shapes include J-shaped, reversed J-shaped, left-skewed, almost symmetric, and right-skewed, with a heavy tail, and extremely high flexible kurtosis, which makes the EGEEx distribution the perfect choice for modeling a wide range of data sets.

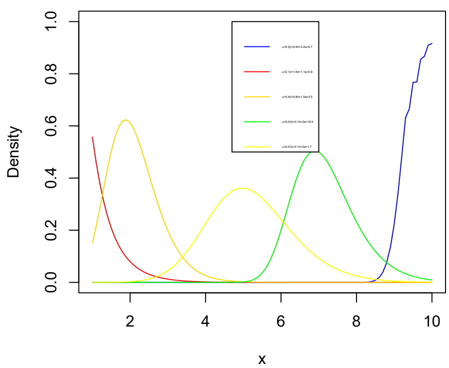


Fig. 1: **The EGEEx density shapes**

From Fig. 2 it is observed that the hazard rate function for different combinations of parameter values shows a multiplicity of shapes, which include increasing, decreasing, bathtub shapes, increasing-decreasing-constant shapes, and decreasing-increasing shapes. The proposed EGEEx distribution has some very appealing hazard rate shapes that make it appropriate for modeling monotonic and non-monotonic hazard behaviors that are more likely to be observed in real-world circumstances such as human mortality, reliability analysis, and biomedical applications by increasing its adaptability to fit different survival data.

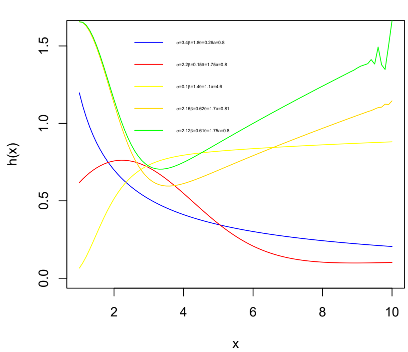


Fig. 2: **The EGEEx Hazard rate function shapes**

B. The EGEEx Sub-Models

Five sub-models of the proposed EGEEx distribution have been considered by letting the parameter(s) of the EGEEx CDF equal one.

1. For $\alpha = a = b = 1$, this reduces the CDF of the EGEEx distribution to the Exponential (Ex) distribution with CDF given as
$F_{Ex}(x) = \left[1 - e^{-\beta x}\right]$, for $\beta > 0, x > 0$.

2. For $\alpha = a = 1$ this reduces the CDF of the EGEEx distribution to the Exponentiated Exponential (EEx) distribution with CDF given as
$F_{EEx}(x) = (1 - e^{-\beta x})^b$, for $b > 0, \beta, x > 0$.

3. For $\alpha = b = 1$, this reduces the CDF of the EGEEx distribution to the Generalized Exponential (GEx) distribution with CDF given as
$F_{GEx}(x) = 1 - \left\{1 - (1 - e^{-\beta x})\right\}^a$, for $a, \beta > 0, x > 0$.

4. For $b = 1$, this reduces the CDF of the EGEEx distribution to the Exponentiated Generalized Exponential (EGEx) distribution with CDF given as
$F_{EGEx}(x) = \left[1 - \left\{1 - (1 - e^{-\beta x})\right\}^a\right]^\alpha$,
for $\alpha > 0, a > 0, \beta > 0, x > 0$.

5. For $\alpha = 1$, this reduces the CDF of the EGEEx distribution to the Generalized Exponentiated Exponential (GEEx) distribution with CDF given as
$F_{GEEx}(x) = 1 - \left\{1 - \left(1 - e^{-\beta x}\right)^b\right\}^a$, for $\beta, a, b > 0, x > 0$.

Table 1: **Summary of the Sub-models**

α	β	a	b	**Sub-model**
1	–	1	1	The Ex distribution
1	–	1	–	The EEx distribution
1	–	–	1	The GEx distribution
–	–	–	1	The EGEx distribution
1	–	–	–	The GEEx distribution

III. STATISTICAL PROPERTIES

The statistical properties of the proposed EGEEx distribution are derived in this section of the paper.

A. *Quantile function*

The median, skewness, and kurtosis have all been computed by researchers using the quantile function, along with other things like conducting simulation studies.
The quantile function of the EGEEx distribution is obtained by inverting CDF shown in Eq. (5). The quantile function is the real solution to the general expression given as

$$F(x) = p, \tag{7}$$

for x is a random variable.
Substituting eq. (5) into Eq. (7) we have

$$= \left[1 - \left\{1 - (1 - e^{-\beta x})^b\right\}^a\right]^\alpha = p \tag{8}$$

Using Mathematica software, eq. (8) becomes

$$x = \frac{1}{\beta} log\left(-\frac{1}{\left(1-\left(1-p^{\frac{1}{\alpha}}\right)^{\frac{1}{a}}\right)^{\frac{1}{b}} - 1}\right) \tag{9}$$

Some quantile function values for different parameters values of α, β, a, and b respectively are shown in **Table 2**. For $p = 0.25, 0.50,$ and 0.75 being the 1^{st} quantile, median, and 3^{rd} quantile respectively.

Table 2: **Summary of some Quantile values of the EGEEx distribution.**

p	**(0.7, 1.2, 0.6, 0.4)**	**(1.7, 0.2,1.5,2.3)**
0.1	0.1319043	0.4326408
0.2	0.3671447	0.7116475
0.25	0.5156701	0.8466258
0.3	0.6851431	0.9830418
0.4	1.0931910	1.2687990
0.50	1.6125590	1.5856390
0.6	2.2841530	1.9548390
0.7	3.1882950	2.4125070
0.75	3.7766800	2.6955750
0.8	4.5084120	3.0369290
0.9	6.8333550	4.0751070

B. *The r^{th} Moments*

The r^{th} Moments general form is given as

$$\rho_r^{'} = \int_0^\infty x^r f(x) dx \tag{10}$$

for $r = 1, 2, ..., n$ and $f(x)$ being the distribution PDF, by substituting Eq. (6) into Eq. 10 gives:

$$\rho_r^{'} = \int_0^\infty x^r \alpha\beta a b e^{-\beta x}\left(1 - e^{-\beta x}\right)^{b-1}\left\{1 - (1 - e^{-\beta x})^b\right\}^{a-1} \left[1 - \left\{1 - (1 - e^{-\beta x})^b\right\}^a\right]^{\alpha-1} dx \tag{11}$$

Binomial expansion representation.

$$(1 - x)^{s-1} = \sum_{k=0}^\infty (-1)^k \binom{s-1}{t} x^k$$

Employing binomial expansion in Eq. (11), we have

$$\left(1 - e^{-\beta x}\right)^{b-1} = \sum_{l=0}^\infty (-1)^l \binom{b-1}{l} (e^{-\beta x})^l,$$

$$\left\{1 - (1 - e^{-\beta x})^b\right\}^{a-1} = \sum_{m=0}^\infty (-1)^m \binom{a-1}{m} (1-e^{-\beta x})^{bm},$$

$$(1 - e^{-\beta x})^{bm} = \sum_{s=0}^\infty (-1)^s \binom{bm}{s} (e^{-\beta x})^s,$$

$$\left[1 - \left\{1 - (1 - e^{-\beta x})^b\right\}^a\right]^{\alpha-1} = \sum_{t=0}^\infty (-1)^t \binom{\alpha-1}{t} \left\{1 - (1 - e^{-\beta x})^b\right\}^{at},$$

$$\left\{1 - (1 - e^{-\beta x})^b\right\}^{at} = \sum_{u=0}^\infty (-1)^u \binom{at}{u} (1-e^{-\beta x})^{bu},$$

$$\left(1 - e^{-\beta x}\right)^{bu} == \sum_{v=0}^\infty (-1)^v \binom{bu}{v} (e^{-\beta x})^v$$

By substituting the binomial expansions above, Eq. (11) leads to;

$$\rho_r^{'} = \alpha\beta ab \sum_{l=0}^\infty \sum_{m=0}^\infty \sum_{s=0}^\infty \sum_{t=0}^\infty \sum_{t=0}^\infty \sum_{u=0}^\infty \sum_{v=0}^\infty \int_0^\infty x^r (-1)^{(l+m+s+t+u+v)} \binom{b-1}{l}\binom{a-1}{m}\binom{bm}{s}\binom{\alpha-1}{t}\binom{at}{u}\binom{bu}{v} \left\{1 - (1 - e^{-\beta x})^b\right\}^{at} \left(1 - e^{-\beta x}\right)^{b(m+u)} \left(e^{-\beta x}\right)^{1+l+s+v} dx \tag{12}$$

By using R software, the first four moments of the EGEEx distribution values for several parameter values are given in **Table 3**. **I:** $\alpha = 2.1$, $\beta = 0.9$, $a = 0.5$, $b = 2.2$; **II:** $\alpha = 21.9$, $\beta = 3.8$, $a = 4.6$, $b = 9.8$; **III:** $\alpha = 14.9$, $\beta = 1.14$, $a = 5.9$, $b = 3$; **VI:** $\alpha = 1.9$, $\beta = 1.1$, $a = 2.9$, $b = 2$.

The EGEEx distribution has a wide range of means and variance, as observed in Table 3 .

The EGEEx distribution skewness values, as shown in Table 3 can be left-skewed ($BS < 0$) and right-skewed ($BS > 0$). The kurtosis values (in Table 3) depict that the EGEEx distribution can be leptokurtic ($MK > 3$), almost mesokurtic ((almost $MK = 3$)), and platykurtic ($MK < 3$).

Table 3: **Summary of r^{th} moments, Bowley skewness (BS), and Moors kurtosis (MK) of the EGEEx**

ρ_r'	I	II	III	IV
ρ_1'	1.5542850	0.0215194	0.0164127	0.2287956
ρ_2'	3.7063216	0.0051762	0.0030818	0.0826628
ρ_3'	12.1028139	0.0012310	0.0005755	0.0414004
ρ_4'	50.6767564	0.0010220	0.0001440	0.0267424
SD	1.1360105	0.0073847	0.0062295	0.1741132
CV	0.7308895	0.3431648	0.3795534	0.7609991
BS	1.5896264	-2.9240020	-2.3847080	1.6322412
MK	6.9927431	0.5059890	3.0941400	7.1770186

C. *Skewness and Kurtosis*

To demonstrate the influence or effect of the additional shape parameters on the measure of shape skewness and kurtosis, the Bowley skewness and Moors kurtosis coefficients are computed using quartiles and octiles, respectively. In spite of the presence of outliers, Bowley skewness, and Moors kurtosis are unaffected by them and are nonetheless guaranteed to exist.

According to [16], the Bowley skewness based on quartiles is given as:

$$BS = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}$$

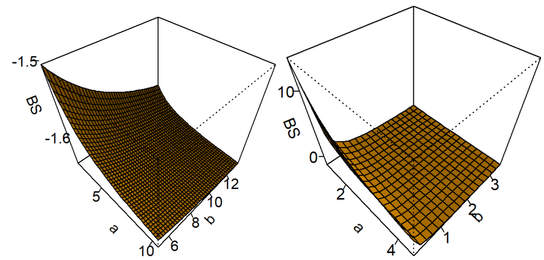
and [17] defines the Moors kurtosis based on octiles as

$$MK = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}$$

for Q is the quantile function.

By using Eq. (9), the Bowley skewness and the Moors kurtosis are obtained.

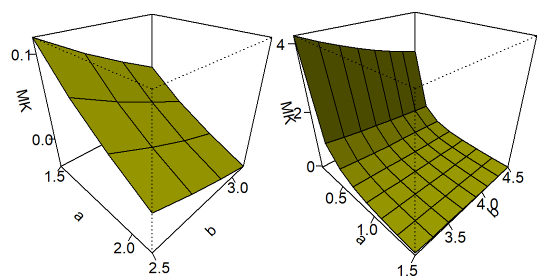
In Fig. 3, Bowley's skewness is depicted in three dimensions (3D) with fixed baseline parameter values of $\alpha = 1.6$ and $\beta = 1.4$. Noticed that the additional parameters affect the shapes of the skewness coefficient. This improves the EGEEx distribution's adaptability and flexibility and supports the significance of the extra parameters.



(a) (b)

Fig. 3: **Plots of Bowley skewness for fixed baseline parameter values of $\alpha = 1.6$ and $\beta = 1.4$.**

In Fig. 4, Moors' kurtosis is depicted in three dimensions (3D) with fixed baseline parameter values of $\alpha = 1.6$ and $\beta = 1.4$. Noticed that the additional parameters affect the shapes of the kurtosis coefficient. This improves the EGEEx distribution's adaptability and flexibility and supports the significance of the extra parameters.



(a) (b)

Fig. 4: **Plots of Moors kurtosis for fixed baseline parameter values of $\alpha = 1.6$ and $\beta = 1.4$.**

D. *Entropy of EGEEx Distribution*

Entropy has been employed as a gauge of the uncertainty or variation of a random variable in a variety of contexts, including science, engineering, probability theory, etc.

As described by [18], the Rényi entropy for a random variable X with any distribution and order φ is given as

$$R_\varphi(X) = \frac{1}{1-\varphi} log \left\{ \int_0^\infty [f(x)]^\varphi \, dx \right\} \tag{13}$$

for $\varphi > 0, \neq 1$.

Substituting Eq. (6) into Eq. (13), we have

$$R_\varphi(X) = \frac{1}{1-\varphi} log \left\{ \int_0^\infty \left(\alpha\beta abe^{-\beta x} \left(1 - e^{-\beta x}\right)^{b-1} wz \right)^\varphi dx \right\} \tag{14}$$

for $w = \left\{1-(1-e^{-\beta x})^b\right\}^{a-1}$, and

$$z = \left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^{\alpha-1}$$

The binomial expansions can be used in Eq. (14) to express the Rényi entropy of the EGEEx distribution, which leads to

$$R_\varphi(X) = \frac{1}{1-\varphi} log\{(\alpha\beta ab)^\varphi \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{t=0}^{\infty} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\int_0^\infty ((-1)^{(l+m+s+t+u+v)}\binom{b-1}{l}\times \binom{a-1}{m}\binom{bm}{s}\binom{\alpha-1}{t}\binom{at}{u}\times \binom{bu}{v}\left\{1-(1-e^{-\beta x})^b\right\}^{at}\left(1-e^{-\beta x}\right)^{b(m+u)} \times \left(e^{-\beta x}\right)^{1+l+s+v})^\varphi dx\} \quad (15)$$

The entropy values of the EGEEx distribution for various parameter values are shown in **Table 4**. The entropy values were obtained through numerical integration using R software by fixing the parameters α, β, a, and b respectively.

Table 4: **Some Rényi entropy values of EGEEx.**

R_φ	**(1.4, 0.2, 1.8, 0.4)**	**(2.7, 1.4, 3.1, 2.9)**
$R_{(0.1)}$	4.52712	0.110187
$R_{(0.6)}$	3.34614	-0.95698
$R_{(1.7)}$	2.93027	-1.31034
$R_{(2.5)}$	2.82319	-1.40174
$R_{(3.4)}$	2.75287	-1.46240

E. *Order Statistic*

The i^{th} order statistic have the general form given as

$$f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \quad (16)$$

The i^{th} order statistic of the EGEEx distribution's PDF is obtained by substituting Eqs. (5) and (6) into Eq.(16), which is given by:

$$f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!}\alpha\beta abe^{-\beta x}\left(1-e^{-\beta x}\right)^{b-1} \left\{1-(1-e^{-\beta x})^b\right\}^{a-1}\left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^{\alpha-1} \left\{\left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^\alpha\right\}^{i-1}\times \left\{1-\left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^\alpha\right\}^{n-i} \quad (17)$$

The PDF of the maximum order statistic, $f_{(n:n)}(x)$ is given as

$$f_{(n:n)}(x) = n\alpha\beta abe^{-\beta x}\left(1-e^{-\beta x}\right)^{b-1}\left\{1-(1-e^{-\beta x})^b\right\}^{a-1}\times \left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^{\alpha-1}\left\{\left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^\alpha\right\}^{n-1} \quad (18)$$

while the PDF of the minimum order statistic, $f_{(1:n)}(x)$ is given as

$$f_{(1:n)}(x) = n\alpha\beta abe^{-\beta x}\left(1-e^{-\beta x}\right)^{b-1}\left\{1-(1-e^{-\beta x})^b\right\}^{a-1}\times \left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^{\alpha-1}\left\{1-\left[1-\left\{1-(1-e^{-\beta x})^b\right\}^a\right]^\alpha\right\}^{n-1} \quad (19)$$

The range, $r_{i,n}(x)$ can be calculated with the help of the maximum and minimum order statistic of the EGEEx distribution, i.e. $r_{i,n}(x) = f_{(n:n)}(x) - f_{(1:n)}(x)$.

IV. PARAMETERS ESTIMATION

This section of the paper presents the maximum likelihood estimations (MLEs) and the simulation study of EGEEx parameters.

A. *Maximum Likelihood Estimation*

Assume an independent random sample of size n, with $X_1, X_2, ..., X_n$ drawn from a distribution, then the likelihood function (ℓ) is given as;

$$\ell = \prod_{i=1}^{n} f(x_i), x>0, f(x_i) = f_{EGEEx}(x_i) \quad (20)$$

The MLEs of the parameters are obtained by simultaneously taking the partial derivatives with respect to the parameters. For the MLEs of the EGEEx distribution parameters α, β, a, and b, substitute (6) into (20) and take the log of likelihood function (ℓ), as shown below:

$$log(\ell) = nlog(\alpha\beta ab) - \beta\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n} log(1-e^{-\beta x_i}) + (a-1)\sum_{i=1}^{n} log\left\{1-(1-e^{-\beta x_i})^b\right\} + (\alpha-1)\sum_{i=1}^{n} log\left[1-\left\{1-(1-e^{-\beta x_i})^b\right\}^a\right] \quad (21)$$

The values that maximize the log-likelihood function presented in Eq. (21) are the maximum likelihood estimates ($\hat{\alpha}, \hat{\beta}, \hat{a}$, and $\hat{b}$) for the parameters ($\alpha.\beta, a$, and b). The partial derivatives of the log-likelihood ($log(\ell)$) function of Eq. (21) with respect to $\alpha.\beta, a$, and b and equating each parameter to zero are as follows:

$$\frac{\partial log(\ell)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left[1-\left\{1-(1-e^{-\beta x_i})^b\right\}^a\right] = 0 \quad (22)$$

$$\frac{\partial log(\ell)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n} x_i\frac{e^{-\beta x_i}}{1-e^{-\beta x_i}} + b(a-1)\sum_{i=1}^{n} x_i\frac{e^{-\beta x}(1-e^{-\beta x_i})^{b-1}}{(1-e^{-\beta x_i})^{b-1}} + a(\alpha-1)b \sum_{i=1}^{n} x_i\frac{e^{-\beta x_i}(1-e^{-\beta x_i})^{b-1}(1-(1-e^{-\beta x_i})^b)^{a-1}}{1-(1-(1-e^{-\beta x_i})^b)^a} = 0 \quad (23)$$

$$\frac{\partial log(\ell)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\left\{1-(1-e^{-\beta x_i})^b\right\} + (\alpha-1)\sum_{i=1}^{n}\frac{\left[1-(1-e^{-\beta x_i})^b\right]^a log\left[1-(1-e^{-\beta x_i})^b\right]}{\left[1-(1-e^{-\beta x_i})^b\right]^a - 1} = 0 \quad (24)$$

$$\frac{\partial log(\ell)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log(1-e^{-\beta x_i}) + (a-1)\sum_{i=1}^{n}\frac{(1-e^{-\beta x_i})^b log(1-e^{-\beta x_i})}{(1-e^{-\beta x_i})^b - 1} + a(\alpha-1)\sum_{i=1}^{n}\frac{(1-e^{-\beta x_i})^b log(1-e^{-\beta x_i})\left(1-(1-e^{-\beta x_i})^b\right)^{a-1}}{1-\left(1-(1-e^{-\beta x_i})^b\right)^a} = 0 \quad (25)$$

The solutions of Eqs. (22)-(25) are found using a numerical optimization method due to the lack of closed-form solutions.

The Broyden-Fletcher-Goldfarb-Shannon (BFGS) algorithm was used in this article to estimate the parameters of the EGEEx distribution, and both the Hessian matrix and the gradient vector of the log-likelihood function were required. The square matrix, called the Hessian matrix, consists of the second-ordered partial derivatives of the log-likelihood function with respect to the parameters. The observed information matrix of the EGEEx distribution is provided as

$$H^{-1}(\phi) = \begin{bmatrix} \frac{\partial^2 log(\ell)}{\partial\alpha^2} & \frac{\partial^2 log(\ell)}{\partial\alpha\partial\beta} & \frac{\partial^2 log(\ell)}{\partial\alpha\partial a} & \frac{\partial^2 log(\ell)}{\partial\alpha\partial b} \\ & \frac{\partial^2 log(\ell)}{\partial\beta^2} & \frac{\partial^2 log(\ell)}{\partial\beta\partial a} & \frac{\partial^2 log(\ell)}{\partial\beta\partial b} \\ & & \frac{\partial^2 log(\ell)}{\partial a^2} & \frac{\partial^2 log(\ell)}{\partial a\partial b} \\ & & & \frac{\partial^2 log(\ell)}{\partial b^2} \end{bmatrix}$$

for $\hat{\phi} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})'$.

B. Simulation study

A Monte Carlo simulation study was carried out on the maximum likelihood estimators for the parameters of the EGEEx distribution to calculate or approximate the average estimate (AE), average bias (AB), mean square error (MSE), and root means square error (RMSE).

The algorithm in the Monte Carlo simulation:

To begin, a random sample of sizes n = 200, 400, ..., 1000 was generated using the quantile function given in Eq. (9), with 1000 iterations for each value of n in the EGEEx distribution.

In the second step, initial parameter values for α, β, a, and b were selected:

Set **I**: (α, β, a, b)=(0.7, 1.3, 1.4, 0.2), and

set **II**: (α, β, a, b)=(1.6, 0.9, 0.8, 1.9).

Finally, the simulation results for each set's average estimate (AE), average bias (AB), mean square error (MSE), and root mean square error (RMSE) are shown in Tables 5 and 6.

To estimate the Average Bias (AB), the Mean Squared Errors (MSE), and the Root Mean Squared Error (RMSE) the following formulas were used

$$AB_{(\psi)} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi}_i - \psi) \quad (26)$$

$$MSE_{(\psi)} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi}_i - \psi)^2 \quad (27)$$

and

$$RMSE_{(\psi)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi}_i - \psi)^2} \quad (28)$$

For N is the number of iterations in each value of n, and $\widehat{\psi}_i$ is an estimator of ψ.

The EGEEx distribution simulation results from Tables 5 and 6 show that the average estimate (AE) converges to the true values of the parameters as the sample size (n) increases, and the AB, MSE, and RMSEs for the parameter estimators decrease as the sample size (n) increases.

Table 5: **Monte Carlo simulation study results**

Set I: (α, β, a, b)=(0.7, 1.3, 1.4, 0.2)								
	AE				**AB**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
200	0.711113	1.294406	1.394701	0.204491	0.011113	-0.005594	-0.005299	0.004491
400	0.704262	1.296734	1.396951	0.202029	0.004262	-0.003266	-0.003049	0.002029
600	0.702241	1.297679	1.397835	0.201661	0.002241	-0.002321	-0.002165	0.001661
800	0.702585	1.298116	1.398244	0.201173	0.002585	-0.001884	-0.001756	0.001173
1000	0.702508	1.298578	1.398667	0.201463	0.002508	-0.001422	-0.001334	0.001463
	MSE				**RMSE**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
200	0.003932	0.000147	0.000132	0.000449	0.062703	0.011214	0.011489	0.021169
400	0.001951	0.000059	0.000052	0.000216	0.044176	0.007701	0.007184	0.014714
600	0.001287	0.000030	0.000026	0.000157	0.035869	0.005511	0.005102	0.012520
800	0.000901	0.000019	0.000016	0.000105	0.030020	0.004366	0.004039	0.010228
1000	0.000692	0.000012	0.000011	0.000059	0.026305	0.003523	0.003298	0.009306

Table 6: **Monte Carlo simulation study results**

set II: (α, β, a, b)=(1.6, 0.9, 0.8, 1.9)								
	AE				**AB**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
200	1.626545	0.903649	0.807350	1.893601	0.026545	0.003649	0.007350	-0.006399
400	1.611138	0.900955	0.804345	1.895433	0.011138	0.000955	0.004345	-0.004566
600	1.610117	0.900978	0.804254	1.896089	0.010117	0.000978	0.004254	-0.003911
800	1.610085	0.900740	0.803364	1.896332	0.010085	0.000740	0.003364	-0.003668
1000	1.608787	0.900638	0.802752	1.896578	0.008787	0.000638	0.002752	-0.003422
	MSE				**RMSE**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
200	0.026561	0.001175	0.001209	0.000685	0.162974	0.034282	0.034773	0.026179
400	0.012420	0.000556	0.000635	0.000321	0.111448	0.023570	0.025195	0.017906
600	0.008913	0.000440	0.000458	0.000226	0.094411	0.020967	0.021407	0.015017
800	0.006692	0.000288	0.000381	0.000150	0.081808	0.016982	0.019525	0.012245
1000	0.005453	0.000255	0.000289	0.000105	0.073847	0.015969	0.017005	0.010246

Remarks on the MLE based on the simulation results

Since an increase in sample size n leads to convergence of the estimates to the true or initial parameter values, and the AB, MSE, and RMSE of the parameters decrease, this shows that the method of parameter estimation using the maximum likelihood method performs well in estimating the parameters of the EGEEx distribution.

V. APPLICATION OF DATA

In this section of the article, a real data set is fitted to the EGEEx distribution and its competing models to examine the importance, adaptability, and flexibility of the EGEEx distribution by using the goodness-of-fit measures; the negative log-likelihood ($-log(\ell)$), Anderson darling (A^*), Cramer-von mises (W^*), Kolmogorov-Smirnov (D) test, Akaike information criteria (AIC), Consistent Akaike information criteria (CAIC), Bayesian information criteria (BIC), and Hannan and quin information criteria (HQIC) .

The goodness-of-fit measures are calculated using the following mathematical formulae;

$$A^* = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{logF(x_i) + log[1-F(x_{n-i+1})]\} \quad (29)$$

$$W^* = \sum_{i=1}^{n}\left(F(x_i) - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n} \quad (30)$$

$$D = max\{|F(x_i) - \widehat{F}(x_i)|, |F(x_i) - \widehat{F}(x_{i-1})|\}, \quad (31)$$

for $i = 1, 2, ..., n$

$$AIC = 2[-log(\ell) + k] \quad (32)$$

$$CAIC = -2log(\ell) + k[log(N) + 1] \quad (33)$$

$$BIC = -2log(\ell) + klog(N) \quad (34)$$

$$HQIC = 2\{-log(\ell) + klog[log(N)]\} \quad (35)$$

where k denotes the number of estimated parameters in the model, ℓ denotes the maximum value of the likelihood function for the distribution, and N denotes the number of sample size/data points.

Data set: An extensive fatigue life of 6061-T6 aluminum coupons

The data comprises of 101 fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second, which are shown in **Table 7**. The same data set was used by [19].

Table 7: **Fatigue life of 6061-T6 aluminum coupons.**

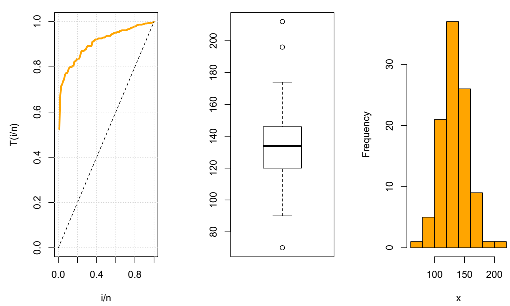
70	90	96	97	99	100	103	104
104	105	107	108	108	108	109	109
112	112	113	114	114	114	116	119
120	120	120	121	121	123	124	124
124	124	124	128	128	129	129	130
130	130	131	131	131	131	131	132
132	132	133	134	134	134	134	134
136	136	137	138	138	138	139	139
141	141	142	142	142	142	142	142
144	144	145	146	148	148	149	151
151	152	155	156	157	157	157	157
158	159	162	163	163	164	166	166
168	170	174	196	212			

Table 8 summarizes the most relevant descriptive statistics of the fatigue life of 6061-T6 aluminum coupons. The skewness value indicates that the data are right-skewed due to its positive sign, and the kurtosis value indicates leptokurtosis characteristics, i.e. the value is greater than 3, which is associated with the normal distribution as observed in Table 8.

Table 8: **Descriptive statistics for the fatigue life of 6061-T6 aluminum coupons.**

Statistic	min.	max.	mean	median	mode	var.	sd.	skewness	kurtosis
Value	70	212	133.7327	133	142	499.7778	22.35571	0.3305	4.0528

The Total Time in Test (TTT)-transform plot from Fig. 5(a) indicates that the data has a concave hazard rate shape (i.e., monotonic increasing) defined by [20] with few outliers.



(a) (b) (c)

Fig. 5: **(a) TTT-transform plot, (b) Boxplot, and (c) Histogram for the data set**

The parameter estimates for the fitted models are shown in Table 9, along with their standard errors in parentheses. All of the parameters of the fitted models are significant at the 5% significance level. The conclusion that all of the parameters of the fitted models are significant at the 5% level of significance can be drawn from the standard error test, which states that a parameter is considered to be significant at the 5% level of significance if the standard error is less than half the parameter value. From Tables 10 and 11, when compared to the competing models, a lower value of goodness-of-fit measures indicate a better fit, and thus the EGEEx distribution fits the fatigue life of 6061-T6 aluminum coupon data better than its competing models, with the exception of the exponential (Ex) distribution, which does not fit due to $p - value < 0.05$.

Table 9: **The estimates and SEs (in parentheses) for the fatigue life of 6061-T6 aluminum coupons.**

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
EGEEx	11.3152(1.9578)	0.1897(0.0694)	1.0860 (0.3982)	0.1033(0.0376)
GEEx	-	6.9862(1.1798)	0.4939(0.0926)	0.0358(0.0067)
EGEx	7.1712(1.0989)	0.0048(0.0007)	3.9106(0.6294)	-
EEx	-	0.0161(0.0013)	-	5.4602(0.8707)
GEx	-	3.2391(0.4669)	0.0023(0.0002)	-
Ex	-	0.0075(0.0007)	-	-

Table 10: **Summary of A^*, W^*, D, and $p - values$ for the fatigue life of 6061-T6 aluminum coupons.**

$Model$	A^*	W^*	D	$p - value$
EGEEx	**0.301**	**0.049**	**0.085**	**0.769**
GEEx	0.444	0.076	0.268	0.724
EGEx	0.440	0.075	0.267	0.731
EEx	0.426	0.072	0.275	0.456
GEx	0.380	0.063	0.490	0.051
Ex	0.382	0.065	0.496	0.003

Table 11: **The $-\log(\ell)$ and information criteria results for the fatigue life of 6061-T6 aluminum coupons.**

Model	$-log(\ell)$	AIC	$CAIC$	BIC	$HQIC$
EGEEx	**501.552**	**1011.104**	**1011.520**	**1021.564**	**1015.338**
GEEx	515.233	1036.467	1036.714	1044.312	1039.643
EGEx	514.313	1034.626	1034.873	1042.471	1037.802
EEx	523.041	1050.083	1050.205	1055.313	1052.200
GEx	595.483	1194.966	1195.088	1200.196	1197.083
Ex	595.480	1192.960	1193.001	1195.575	1194.019

The fitted densities plot for the EGEEx and its sub-models in Fig. 6 shows that the EGEEx distribution best fit or mimic the fatigue life of 6061-T6 aluminum coupon data than its sub-models.

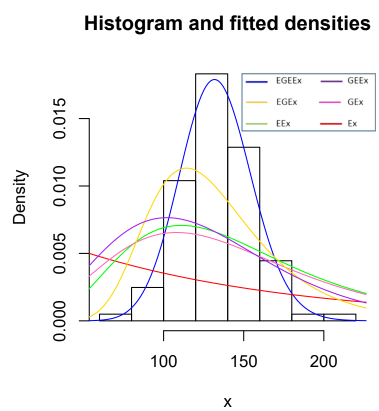


Fig. 6: **Fitted densities for the fatigue life of 6061-T6 aluminum coupons**

VI. CONCLUSION

When it comes to the most commonly used baseline distribution by mathematicians and statisticians in the modification or extending distribution then the exponential distribution is no exception. This article proposes the exponentiated generalized exponentiated exponential distribution (written as EGEEx). Some basic statistical properties of the proposed EGEEx distribution are investigated, including the hazard rate function, quantile function, moments, entropy, and order statistic. The maximum likelihood estimators of the EGEEx parameters are derived, and a Monte Carlo simulation analysis was conducted to compute the asymptotic properties of the MLEs, the average estimates, the average bias, the mean square error, and the root mean square error. The simulation analysis shows that the estimates converge to the true value of the parameter as the sample size increases, and the values of the average biases, the mean square errors, and the root mean square errors decrease with an increase in the sample size. Finally, the flexibility and adaptability of the EGEEx distribution were tested by applying the fatigue life of the 6061-T6 aluminum coupon data set, and the results show that the EGEEx distribution is flexible enough to analyze data that exhibit heavy tails and that the EGEEx distribution performs better than its competing models in analyzing the data set.

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