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Application of the Least Action Principle inScience and Engineering

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Abstract:

Physics, fluids, transport, circuit are the core curriculum across engineering discipline. Students often isolate concepts and have difficulty to foresee the interrelations and analogy between these fundamental phenomena. Inspired by the development of the least action principle, this study shows the uniform minimization approach to derive two physics rules (Newton's law for Force, Snell's law) and three related examples from engineering (flow of current in a circuit, fluid flow and heat transfer by conduction). This approach will allow the student to better understand the application of minimization rule in engineering concepts and provide an easy tool for the instructor to convey the concepts across the engineering curriculum.

Keywords, Least action, electrical circuits, fluid flow, heat transfer, pressure drop

I. INTRODUCTION

An interesting problem not explored in this paper, which is of historical significance is called the Brachistochrone, which means the shortest time required (to travel). Johann Bernoulli and Isaac Newton were among the first to solve the problem of the quickest descent of a particle in a gravitational field from a given point to another, which was not directly below it. The path that signifies the shortest travel time is not a straight line but it is given by a cycloid curve. The developments since then paved way for many branches in mathematics, including calculus of variation and also of the development of the principle of least action (1).

The principle of the least action principle was founded many centuries ago. Its development involved contributions from Fermat to Feynman as summarized in a review paper by Ogborn et al (2). In simple words, it implies that the path taken by an entity is such that the action involved is minimized. The entity could be a projectile or electric current or fluid flow or heat flow by conduction. The path used in the above definition also applies to the distribution of the entity over a given geometry.

$$S = \int_{initial\ event}^{final\ event} Ldt = \int_{initial\ event}^{final\ event} (KE - PE)dt \qquad (1)$$

Where the Lagrangian (L) is expressed in terms of the difference between the kinetic energy (KE) and potential energy (PE). In a recent paper, Ohara and Yamatani (3) obtained an expression for the cross-section area of a channel, which is the most efficient for the case of minimum wetting area for a given flow using a method, which is in conformity with the principle of least action. Sunil Nath (4) has extended the applicability of the least action principle to biological systems, specifically by analyzing the experimental data on the rat liver mitochondria with succinate as a substrate. Lezhniuk et al (5) have applied the principle using power minimization as a criterion for determining the self-optimization modes of electric grids with renewable energy sources. It leads to minimum power losses during transmission. Similar to the approach used for solving the Brachistochrone problem, Su et al (6) derived appropriate expressions for the constant water head horizontal absorption and vertical infiltration for a special case involving water-retention curve and power law expression for hydraulic conductivity. Zhao et al (7) have studied the optimization of reversible thermodynamic processes and Carnot cycles. Sree Harsha (8) has used the principle to demonstrate how to solve for distribution of current in DC circuits. Hua and Guo (9) have outlined the application in terms of heat transfer by conduction. They showed that for one-dimensional heat flow the optimal temperature distribution is in accordance with least destruction of heat transfer ability and determined the optimal insulation thickness distribution.

II. ANALYSIS

For any physical system, it has been observed that an object chooses a path that minimizes the action. It can be shown that the extrema of action occur as given by the following Euler-Lagrange Equation:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{dL}{d\dot{q}} = 0 \qquad (2)$$

Where q re and $\dot{q}$ represent the distance and the velocity, respectively. This paper will explore the application of above principle to two examples from physics (Newton's law for force, Snell's law) and three related examples from engineering (flow of current in a circuit, fluid flow and heat transfer by conduction).

A. Derivation of Equation for Newton's Law for Force

In mechanics, the expression for force which is written as the product of mass (m) and acceleration (a or $\ddot{x}$) can be derived using the above expression.

$$L = KE - PE = \frac{1}{2}m\dot{x}^2 - U(x)$$

$\frac{\partial L}{\partial x} = 0 - \frac{\partial U(x)}{\partial x}$; and $\frac{d}{dt}\frac{dL}{d\dot{x}} = \frac{d}{dt}(m\dot{x})$ -0=m$\ddot{x}$

When substituted in equation (2), we obtain the following:

$$F = -\frac{\partial U(x)}{\partial x} = m\ddot{x} \qquad (3)$$

Thus, the expression for force can be derived from the least action principle. The other interpretation is that the laws of Newton can be expressed as: the path taken by an object that it implies the difference between the average kinetic energy and the average potential energy is as small as possible.

B. Refraction of Light

One of the exercises listed in college calculus texts has to do with investigation of the principle of time in optics, which is named after French mathematician Pierre de Fermat. It can be shown mathematically that the path taken between any two points in a medium by a ray of light is such that the time required for its travel is minimum.

Consider the propagation of a ray of light from medium 1 to medium 2 as shown in Figure 1.

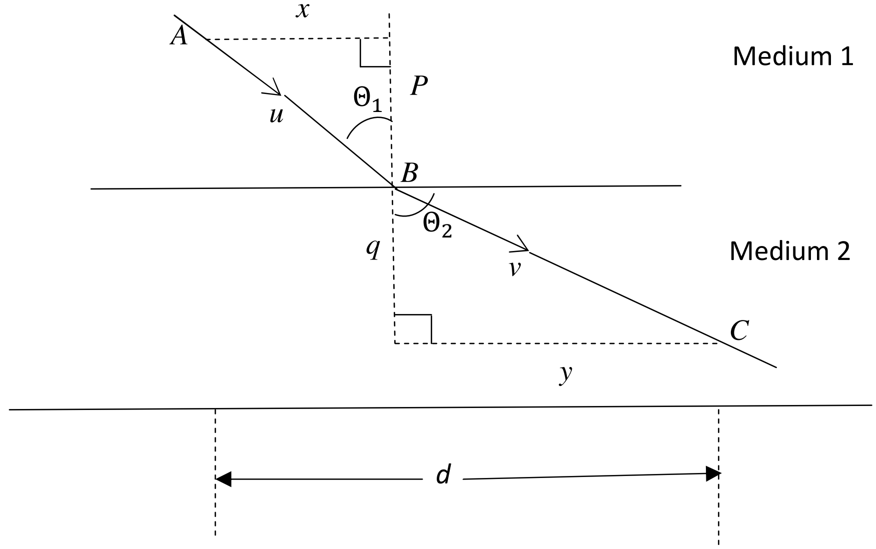

Figure 1 Refraction of light as it travels from one medium to another

The time t taken for the light to travel is given by the sum of the individual time taken to travel in medium 1 at velocity of u and medium 2 at velocity of v, which is found by dividing the distance and dividing it by the speed, thus:

$$t = \frac{1}{u}\sqrt{x^2 + P^2} + \frac{1}{v}\sqrt{y^2 + q^2}$$

But

$$y = d - x$$

Therefore,

$$t = \frac{1}{u}\sqrt{x^2 + P^2} + \frac{1}{v}\sqrt{(d - x)^2 + P^2}$$

In order to minimize the time of travel, differentiate t in the above equation with respect to x and equate it to zero.

$$\frac{dt}{dx} = \frac{1}{2}\frac{2}{u}\frac{x}{\sqrt{x^2 + P^2}} + \frac{1}{2}\frac{2}{v}\frac{(d-x)(-1)}{\sqrt{(d-x)^2 + q^2}} \qquad (A)$$

As a first condition for the minima, equate equation (A) to zero, which gives us:

$$\frac{1}{u}\frac{x}{\sqrt{x^2 + P^2}} = \frac{1}{v}\frac{(d-x)}{\sqrt{(d-x)^2 + P^2}}$$

$$\frac{1}{u}\sin\theta_1 = \frac{1}{v}\sin\theta_2$$

Multiply both sides of the equation with C, the speed of light in a vacuum.

$$\frac{C}{u}\sin\theta_1 = \frac{C}{v}\sin\theta_2$$

i.e.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Where n is the refraction index of the medium, which is defined as the ratio of the speed of light in a vacuum to the phase velocity of light in the medium. The above equation is known as Snell's Law.

The second order condition is obtained by differentiating equation (A) with respect to time the distance, x, which leads to the following:

$$\frac{1}{u}\frac{Cos^2\theta_1}{\sqrt{x^2+P^2}} + \frac{1}{v}\frac{Cos^2\theta_2}{\sqrt{(d-x)^2+P^2}} \text{(B)}$$

It can be easily shown that the above expression is always greater than zero. Thus, result obtained in equation (B) corresponds to the fact that the path taken by light as it travels from one medium to the other is such that the time of travel is minimized.

Using the a principle outlined above, an attempt is made in this paper to mathematically explain the principle of the distribution of electrical current, the heat flow by conduction, and the fluid (volumetric rate) flow when they encounter parallel resistances under steady state conditions. The following analysis is an attempt to show that in all the three cases considered, the flow (current, heat, fluid) occurs in a way that the rate of energy dissipation is minimized.

C. Current Flow in an Electric Circuit

As an example, let us consider the electrical current, heat flow and fluid flow distribution through two parallel pathways as shown in Figure 2-a, 2-b, and 2-c respectively. We will first examine the current flow (Figure 2-a) where the total rate of energy dissipation, i.e. power (PW) can be written as the sum of the power dissipated through the two resistances, R_1 and R_2.

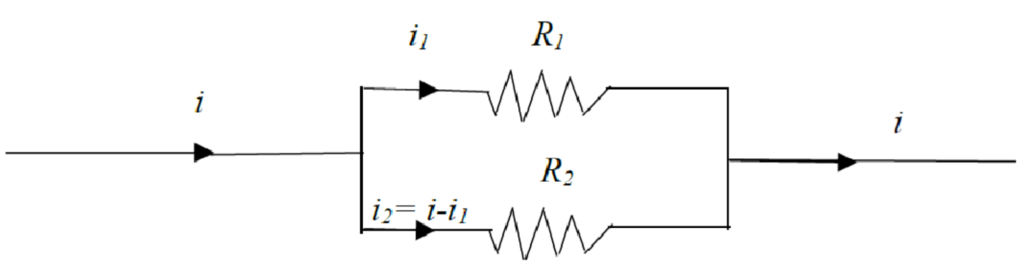

Figure 2-a Electrical current flow through two resistance in parallel.

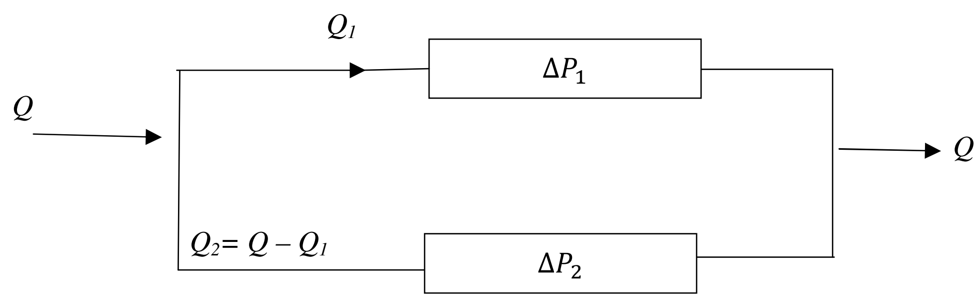

Figure 2-b Fluid Flow through two parallel pipes.

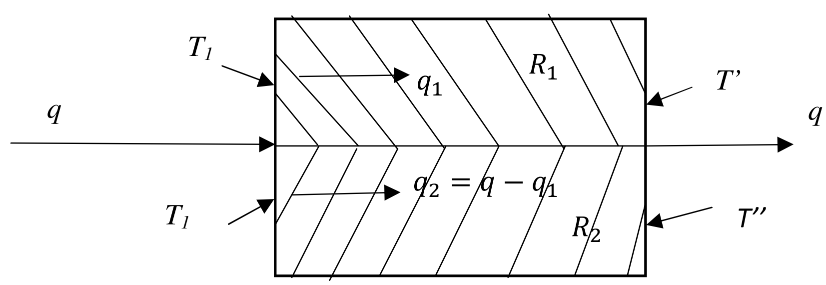

Figure 2-C, Heat Transfer by conduction in two resistances in parallel

$$PW = {i_1}^2 R_1 + (i - i_1)^2 R_2 \qquad (1)$$

Where i and i_1 represent the total current and the current flow through R_1, respectively.

The first differential of P with respect to i_1 gives us:

$$\frac{dPW}{di_1} = 2i_1R_1 + 2(i - i_1)(-1)R_2 \qquad (2)$$

Equating the above equation to zero give us:

$$i_1R_1 = (i - i_1)R_2(3)$$

Which implies that the voltage drop across the resistance R_1 equals toR_2 in conformity with the Kirchhoff's circuit law. Differentiating Equations (2) with $i_1$gives us:

$$\frac{d^2PW}{di_1^2} = 2[R_1 + R_2] > 0 \qquad (3-a)$$

From the first and second conditions, for differentiation, we observe that equation (3) implies that the current flow through a circuit occurs in a way that it minimizes the rate of energy dissipation.

Let us multiply both sides of equation (3) by the voltage drop V across the resistors and noting that: $i_2 = (i - i_1)$

$$\mathrm{V}i_1R_1 = Vi_2R_2(4)$$

Since the voltage multiplies by the current represents the power dissipated, we conclude that in this case:

$$PW_1R_1 = PW_2R_2(5)$$

D. Fluid Flow: Parallel Pipes

In the study of fluid mechanics, one learns that the pressure drop across two parallel elements (say pipes) is equal. Again, the principle of total power dissipation can be employed. Let Q represent the total volumetric flow rate while Q_1 is the flow rate through the element characterized by the element that encounters pressure dropΔP_1. Thus, the total power dissipation will be given by:

$$PW = Q_1 \Delta P_1 + (Q - Q_1)\Delta P_2 \qquad (4)$$

The pressure drop ΔP , which is related to the length (L), diameter of the pipe (D), is given by the following expression:

$$\Delta P = f \frac{L}{D}\frac{v^2}{2} \qquad (5)$$

Where v is the linear velocity. The friction factor$f$ is proportional to the Reynolds number (Re). For laminar and turbulent flow, the friction factor is given by the following expressions:

$$f = \frac{a}{Re} \qquad \textit{Laminar flow}, \text{ and}$$

$$f = b\, Re^{-0.2} \qquad \textit{tubulent flow}$$

Thus, in general we can express the friction factor as

$$f = C_3 {R_e}^{-n} \qquad 0 < n \leq 1 \qquad (6)$$

Since the volumetric flow rate is equal to the linear velocity times the cross-sectional area of the flow, we can rewrite equation (5) as:

$$\Delta P = C v^{2-n} \qquad (8)$$

Where the constant C is given by (for circular tubes):

$$C = \frac{1}{2}\left(\frac{4}{\pi}\right)^{2-n}\left(\frac{\rho}{\mu}\right)^{-n} D^{-(3-n)} > 0 \qquad (9)$$

Thus, equations (4) can be expressed as:

$$PW = C_1 {Q_1}^{3-n} + C_2 {Q_2}^{3-n} \qquad (10)$$

Differentiating the above expression with respect to v_1 and equating to zero give us:

$$\frac{dPW}{dQ_1} = (3-n)C_1 {Q_1}^{2-n} + (3-n)(-1)C_2 {Q_2}^{2-n} = 0 \quad (10-a)$$

Which gives us:

$$C_1 {Q_1}^{2-n} = C_2 {Q_2}^{2-n} \qquad (10-b)$$

Substitution of (10-b) in (8) gives us:

$$\Delta P_1 = \Delta P_2 \qquad (11)$$

The second order condition is obtained by differentiating equation (10-a) again w.r.tQ_1

$$\frac{d^2P}{d{Q_1}^2} = (3-n)(2-n)\left[C_1 {Q_1}^{1-n} + C_2 {Q_2}^{1-n}\right]$$

Since,$0 < n \leq 1$,

$$\frac{d^2P}{d{Q_1}^2} > 0 \qquad \text{(11-a)}$$

Hence, the above equation represents the condition for the minimization of the rate of energy dissipation during the fluid flow.

E. Heat Conduction: Parallel Walls

Now consider heat flow (Figure 2-C). We can use the result obtained from our analysis of electrical systems that the product of power and resistance across each element must be equal. In the case of heat transfer power is represented by the rate of heat flow (q), thus, we have,

$$q_1 R_1 = q_2 R_2 \qquad (12)$$

Since the rate of heat transfer under steady state conditions can be represented as the ratio of the driving force (temperature difference across the element) to the resistance, equation (12) gives us:

$$(\frac{\Delta T'}{R_1})R_1 = (\frac{\Delta T''}{R_2})R_2 \qquad (13)$$

$$\Delta T' = \Delta T'' \qquad (13)$$

Thus, the driving force when the when the two resistances are in parallel are equal to one another.

III. CONCLUSION

This study demonstrated a simple minimization approach to derive two physics rules (Newton's law for Force and the Snell's law) and three related examples from engineering (flow of current in a circuit, fluid flow and heat transfer by conduction). The approach adopted involved finding the first order and second order derivatives of the underlying function. In addition, the methods offer testimony to the fact that transport phenomena whether they relate to the electrical current, heat flow or fluid flow point to the similar mathematical and physical formulation. This derivation provides an effective teaching tool for engineering transport concepts.

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