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New Integral inequalities Via Harmonically Exponential Type s–convex Functions

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Abstract

In this work, we define and investigate the idea and concept of harmonically exponential type s–convex function. We elaborate the newly introduced idea with examples and some nice algebraic properties. Using the newly introduced idea, we attain the new version of Hermite–Hadamard inequality. As a result, several new integral inequalities mean that refinements of Hermite–Hadamard inequality will be established regarding this newly introduced idea.

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1 Introduction

The theory of convexity presents an attractive and absorbing field of research activities and also played a magnificent role in different areas of pure and applied sciences. This theory is important in theoretical aspects of mathematics and also for economists and physicists. Using this theory, mathematicians provide the solution of problems that arise in mathematics. Due to a lot of importance, this theory has become a rich source of inspiration and absorbing field for researchers. Interested readers are referred to [1, 2, 3, 4]. This theory also played a meaningful role in the development of the theory of inequalities. Both the theory of convexity and the theory of inequalities are closely related to each other. In approximation theory, integral inequalities explain and brief the growth rate of competing for mathematical analysis. Thus a rich and meaningful literature on inequalities can be found for the convexity, see [5, 6, 7, 8, 9, 21].

The aim of this research article is to define and introduce a new class of functions called harmonically exponential type s–convex function and also to study some of their nice algebraic properties. Several new inequalities via harmonically exponential type s–convexity will establish. Examples with logic and applications via newly introduce definition will make. Before we start, we need the following necessary known definitions and literature.

2 Preliminaries

Definition 2.1 *[10] Let $\mathbb{Q}: I \to \mathbb{R}$ be a real valued function. A function $\mathbb{Q}$ is said to be convex, if*

$$\mathbb{Q}(\omega\eta_1 + (1-\omega)\eta_2) \leq \omega\mathbb{Q}(\eta_1) + (1-\omega)\mathbb{Q}(\eta_2), \tag{2.1}$$

holds for all $\eta_1, \eta_2 \in I$ and $\omega \in [0,1]$.

Any paper on Hermite inequalities seems to be incomplete without mentioning the well–known Hermite–Hadamard inequality (see [11]) which states, if $\mathbb{Q}: I \to \mathbb{R}$ is a convex function for all $\eta_1, \eta_2 \in I$, then

$$\mathbb{Q}(\frac{\eta_1+\eta_2}{2}) \leq \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} \mathbb{Q}(x)dx \leq \frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}. \tag{2.2}$$

Turkish mathematician İ. İşcan (see [12]) first time defined harmonic convex function in the published paper namely “Hermite–Hadamard type inequalities for harmonically convex functions" which is defined by

Definition 2.2 *A function $\psi: H \subseteq (0,+\infty) \to \mathbb{R}$ is known as harmonic convex, if*

$$\psi(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \leq \omega\psi(\eta_1) + (1-\omega)\psi(\eta_2), \tag{2.3}$$

holds for all $\eta_1, \eta_2 \in H$ and $\omega \in [0,1]$.

Theorem 2.1 *[12] Let $\mathbb{Q}: H \subseteq (0,+\infty) \to \mathbb{R}$ be a harmonically convex function. If $\mathbb{Q} \in L[\eta_1, \eta_2]$ for all $\eta_1, \eta_2 \in H$ with $\eta_1 < \eta_2$, then*

$$\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}) \leq \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} \frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha \leq \frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}. \tag{2.4}$$

Definition 2.3 *[13] A nonnegative function $\mathbb{Q}: I \to \mathbb{R}$ is known to be exponentially convex function if*

$$\mathbb{Q}(\omega\eta_1 + (1-\omega)\eta_2) \leq (e^{\omega}-1)\mathbb{Q}(\eta_1) + (e^{1-\omega}-1)\mathbb{Q}(\eta_2). \tag{2.5}$$

for all $\eta_1, \eta_2 \in I$ and $\omega \in [0,1]$.

The class of all exponentially type convex function on the interval J is indicated by $EXPC(I)$. For the applications and amazing literature about exponential type convex function, see [14, 15, 16, 17, 18].

Definition 2.4 *[19] A function $\varphi: [0,+\infty) \to \mathbb{R}$ is called s–convex in the second sense, if*

$$\mathbb{Q}(\omega\eta_1 + (1-\omega)\eta_2) \leq \omega^s\mathbb{Q}(\eta_1) + (1-\omega)^s\mathbb{Q}(\eta_2) \tag{2.6}$$

holds, $\forall\eta_1, \eta_2 \in [0,+\infty)$, $s \in (0,1]$ and $\omega \in [0,1]$.

Owing to the aforementioned trend and energized by the in-progress research activities in this captivating field, we organize the paper in the following pattern, First of all we will give in Section 3, the idea and its algebraic properties of harmonically exponential type s–convex function. In Section 4, we will derive the new version of Hermite–Hadamard inequality by using the newly introduced definition. As a result in Section 5, we will give related results and Finally, a brief conclusion will be provided as well.

3 .Algebraic properties of harmonically exponential type s–convex function

In this section, we are to add and going to introduce a new definition namely harmonically exponential type s–convex function.

Definition A nonnegative real valued function $\mathbb{Q}: H \subseteq (0,+\infty) \to [0,+\infty)$ is called harmonically exponential type s–convex, if

$$\mathbb{Q}(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \leq (e^{s\omega}-1)\mathbb{Q}(\eta_1) + \left(e^{s(1-\omega)}-1\right)\mathbb{Q}(\eta_2) \quad (3.1)$$

holds for every $\eta_1, \eta_2 \in H$, $s \in [\ln 2.5, 1]$ and $\omega \in [0,1]$.

Remark 3.1 *Taking $s = 1$ in Definition 3, we obtain the following new definition about harmonically exponential type convex function:*

$$\mathbb{Q}(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \leq (e^{\omega}-1)\mathbb{Q}(\eta_1) + (e^{1-\omega}-1)\mathbb{Q}(\eta_2). \quad (3.2)$$

Lemma 3.1 *The following inequalities $e^{s\omega}-1 \geq \omega$ and $e^{s(1-\omega)}-1 \geq (1-\omega)$ are hold. If for all $\omega \in [0,1]$ and $s \in [ln 2.5, 1]$.*

Proof. The proof is evident.

Proposition 3.1 *Let $I \subset (0,+\infty)$ be a harmonically convex set. Every harmonically convex function on a harmonically convex set is harmonically exponential type s–convex function.*

Proof. Using the definition of harmonically convex function and from the Lemma 3.1, we have

$$\mathbb{Q}(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \leq \omega\mathbb{Q}(\eta_1) + (1-\omega)\mathbb{Q}(\eta_2)$$

$$\leq (e^{s\omega}-1)\mathbb{Q}(\eta_1) + \left(e^{s(1-\omega)}-1\right)\mathbb{Q}(\eta_2).$$

This mean that, the new class of harmonically exponential type s–convex function is very larger with respect the known class of functions, like convex and harmonically convex. This is an advantage of the proposed new Definition 3.

Now we makes some examples via newly introduce definition harmonically s–type convex function.

Example 3.2 $\mathbb{Q}(\alpha) = \alpha^2 e^{\alpha^2}$ *is non-decreasing convex function on $(0,1)$, so it is harmonic convex function (see [1]). By using Proposition 3.1, it is harmonically exponential type s–convex function.*

Example 3.3 $\mathbb{Q}(\alpha) = e^{\alpha}$ *is increasing convex function, so it is harmonic convex function (see [1]). By using Proposition 3.1, it is harmonically exponential type s–convex function.*

Example 3.4 $\mathbb{Q}(\alpha) = sin(-\alpha)$ *is non-decreasing convex function on $(0, \frac{\pi}{2})$, so it is harmonic* *convex function* $\forall\alpha \in (0,\frac{\pi}{2})$ *(see [1]). By using Proposition 3.1, it is harmonically exponential type s–convex function.*

Example 3.5 $\mathbb{Q}(\alpha) = \alpha$ *is non-decreasing convex function for* $\alpha \geq 0$, *so it is harmonic convex function* $\forall\alpha \in (0,\infty)$ *(see [1]). By using Proposition 3.1, it is harmonically exponential type s–convex function.*

Example 3.6 $\mathbb{Q}(\alpha) = ln\alpha$ *is harmonic convex function on* $(0,\infty)$ *(see [1]. By using Proposition 3.1 , we get* $\mathbb{Q}(\alpha)$ *is harmonically exponential type s–convex function.*

Example 3.7 $\mathbb{Q}(\alpha) = \sqrt{\alpha}$ *is harmonic convex function for nonnegative value of x. By using Proposition 3.1 , we get* $\mathbb{Q}(\alpha)$ *is harmonically exponential type s–convex function.*

Now, we will study some of their algebraic properties.

Theorem 3.8 *Let* $\mathbb{Q}_1, \mathbb{Q}_2 : H \subseteq (0,+\infty) \to [0,+\infty)$. *If* $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *are two harmonically exponential type convex functions, then*

1. $\mathbb{Q}_1 + \mathbb{Q}_2$ is harmonically exponential type s–convex function;
2. For nonnegative real number $c, c\mathbb{Q}$ is harmonically exponential type s–convex function.

Proof.(1) Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be harmonic s–type convex, then

$$\begin{aligned}
&(\mathbb{Q}_1 + \mathbb{Q}_2)(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \\
&= \mathbb{Q}_1(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) + \mathbb{Q}_2(\frac{\eta_1\eta_2}{\omega\eta_2+(1-t)\eta_1}) \\
&\leq (e^{s\omega}-1)\mathbb{Q}_1(\eta_1) + \left(e^{s(1-\omega)}-1\right)\mathbb{Q}_1(\eta_2) \\
&+(e^{s\omega}-1)\mathbb{Q}_2(\eta_1) + \left(e^{s(1-\omega)}-1\right)\mathbb{Q}_2(\eta_2) \\
&= (e^{s\omega}-1)[\mathbb{Q}_1(\eta_1)+\mathbb{Q}_2(\eta_1)] + \left(e^{s(1-\omega)}-1\right)[\mathbb{Q}_1(\eta_2)+\mathbb{Q}_2(\eta_2)] \\
&= (e^{s\omega}-1)(\mathbb{Q}_1+\mathbb{Q}_2)(\eta_1) + \left(e^{s(1-\omega)}-1\right)(\mathbb{Q}_1+\mathbb{Q}_2)(\eta_2).
\end{aligned}$$

(2) Let $\mathbb{Q}$ be harmonically exponential type s–convex function, then

$$\begin{aligned}
&(c\mathbb{Q})(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}) \\
&\leq c[(e^{s\omega}-1)\mathbb{Q}(\eta_1) + \left(e^{s(1-\omega)}-1\right)\mathbb{Q}(\eta_2)] \\
&= (e^{s\omega}-1)c\mathbb{Q}(\eta_1) + \left(e^{s(1-\omega)}-1\right)c\mathbb{Q}(\eta_2) \\
&= (e^{s\omega}-1)(c\mathbb{Q})(\eta_1) + \left(e^{s(1-\omega)}-1\right)(c\mathbb{Q})(\eta_2),
\end{aligned}$$

which completes the proof.

Theorem 3.9 *Let* $\mathbb{Q}_1 : H \to [0,+\infty)$ *be harmonically convex function and* $\mathbb{Q}_2 : [0,+\infty) \to [0,+\infty)$ *is non-decreasing and exponential type s–convex function. Then the function* $\mathbb{Q}_2 \circ \mathbb{Q}_1 : H \to [0,+\infty)$ *is harmonically exponential type s–convex.*

Proof. For all $\eta_1, \eta_2 \in H$, and $\omega \in [0,1]$, we have

$$\begin{aligned}
&(\mathbb{Q}_2 \circ \mathbb{Q}_1)\left(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}\right) \\
&= \mathbb{Q}_2\left(\mathbb{Q}_1\left(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}\right)\right) \\
&\leq \mathbb{Q}_2\big(\omega\mathbb{Q}_1(\eta_1) + (1-\omega)\mathbb{Q}_1(\eta_2)\big) \\
&\leq (e^{s\omega}-1)\mathbb{Q}_2\big(\mathbb{Q}_1(\eta_1)\big) + \big(e^{s(1-\omega)}-1\big)\mathbb{Q}_2\big(\mathbb{Q}_1(\eta_2)\big) \\
&= (e^{s\omega}-1)(\mathbb{Q}_2 \circ \mathbb{Q}_1)(\eta_1) + \big(e^{s(1-\omega)}-1\big)(\mathbb{Q}_2 \circ \mathbb{Q}_1)(\eta_2),
\end{aligned}$$

which completes the proof.

Theorem 3.10 *Let $0 < \eta_1 < \eta_2, \mathbb{Q}_j : [\eta_1, \eta_2] \to [0, +\infty)$ be a family of harmonically exponential type s–convex functions and $\mathbb{Q}(u) = sup_j \mathbb{Q}_j(u)$. Then $\mathbb{Q}$ is an harmonically exponential type s–convex function and $U = \{u \in [\eta_1, \eta_2] : \mathbb{Q}(u) < +\infty\}$ is an interval.*

Proof. Let $\eta_1, \eta_2 \in U$ and $\omega \in [0,1]$, then

$$\begin{aligned}
&\mathbb{Q}\left(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}\right) \\
&= \sup_j \mathbb{Q}_j\left(\frac{\eta_1\eta_2}{(\omega\eta_2+(1-\omega)\eta_1)}\right) \\
&\leq \frac{1}{n}\sum_{i=1}^{n} (e^{s\omega}-1)\sup_j \mathbb{Q}_j(\eta_1) + \big(e^{s(1-\omega)}-1\big)\sup_j \mathbb{Q}_j(\eta_2) \\
&= (e^{s\omega}-1)\mathbb{Q}(\eta_1) + \big(e^{s(1-\omega)}-1\big)\mathbb{Q}(\eta_2) < +\infty,
\end{aligned}$$

which completes the proof.

Theorem 3.11 *If the function $\mathbb{Q} : [\eta_1, \eta_2] \subseteq (0, +\infty) \to [0, +\infty)$ is harmonically exponential type s–convex, then $\mathbb{Q}$ is bounded on $[\eta_1, \eta_2]$.*

Proof. Let $L = \max\{\mathbb{Q}(\eta_1), \mathbb{Q}(\eta_2)\}$ and $\alpha \in [\eta_1, \eta_2]$ be an arbitrary point. Then there exists $\omega \in [0,1]$ such that $\alpha = \frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}$. Thus, since $e^{s\omega} \leq e$ and $e^{s(1-\omega)} \leq e$, we have

$$\begin{aligned}
\mathbb{Q}(\alpha) &= \mathbb{Q}\left(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}\right) \\
&\leq (e^{s\omega}-1)\mathbb{Q}(\eta_1) + \big(e^{s(1-\omega)}-1\big)\mathbb{Q}(\eta_2) \\
&\leq (e^{s\omega}-1)L + \big(e^{s(1-\omega)}-1\big)L \\
&\leq (e-1) = M.
\end{aligned}$$

We have shown that $\mathbb{Q}$ is bounded above from real number M. Interested reader can also prove the fact that $\mathbb{Q}$ is bounded below using the same idea as in Theorem 2.4 in [13].

Remark 3.2 *Interested readers can find many other nice properties of this new class of functions. We omit here the details.*

4 Hermite–Hadamard type inequality via harmonically exponential type s–convex functions

The purpose of this section is to derive and find a new version of Hermite–Hadamard type inequality via newly introduced definition namely harmonically exponential type s–convexity.

Theorem 4.1 *Let $\mathbb{Q}:[\eta_1,\eta_2]\to[0,+\infty)$ be an harmonically exponential type s–convex function. If $\mathbb{Q}\in L[\eta_1,\eta_2]$, then*

$$\frac{1}{2(\sqrt{e^s}-1)}\mathbb{Q}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\le\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha \tag{4.1}$$

$$\le\left(\frac{e^s-s-1}{s}\right)[\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)].$$

Proof. Since $\mathbb{Q}$ is harmonically exponential type s–convex function, we have

$$\mathbb{Q}\left(\frac{\alpha\beta}{\omega\beta+(1-\omega)\alpha}\right)\le(e^{s\omega}-1)\mathbb{Q}(\alpha)+\left(e^{s(1-\omega)}-1\right)\mathbb{Q}(\beta),$$

which lead to

$$\mathbb{Q}\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\le\left(\sqrt{e^s}-1\right)\mathbb{Q}(\alpha)+\left(\sqrt{e^s}-1\right)\mathbb{Q}(\beta).$$

Using the change of variables, we get

$$\mathbb{Q}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\le\left(\sqrt{e^s}-1\right)\left[\mathbb{Q}\left(\frac{\eta_1\eta_2}{(\omega\eta_2+(1-\omega)\eta_1)}\right)+\mathbb{Q}\left(\frac{\eta_1\eta_2}{(\omega\eta_1+(1-\omega)\eta_2)}\right)\right].$$

Integrating with respect to ω on $[0,1]$ the above inequality, we obtain

$$\frac{1}{2(\sqrt{e^s}-1)}\mathbb{Q}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\le\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha,$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $\alpha=\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}$ and using Definition 3 for the function $\mathbb{Q}$, we have

$$\begin{aligned}
&\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha\\
&=\int_0^1\mathbb{Q}\left(\frac{\eta_1\eta_2}{\omega\eta_2+(1-\omega)\eta_1}\right)d\omega\\
&\le\int_0^1\left[(e^{s\omega}-1)\mathbb{Q}(\eta_1)+\left(e^{s(1-\omega)}-1\right)\mathbb{Q}(\eta_2)\right]d\omega\\
&=\mathbb{Q}(\eta_1)\int_0^1(e^{s\omega}-1)d\omega+\mathbb{Q}(\eta_2)\int_0^1\left(e^{s(1-\omega)}-1\right)d\omega\\
&=\left(\frac{e^s-s-1}{s}\right)[\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)],
\end{aligned}$$

which completes the proof.

Remark 4.1 *Choosing $s=1$ in Theorem 4.1, then we get the Corollary 1 in [6].*

5 New inequalities for harmonically exponential type s–convex function

The principal intention of this section is to derived the refinements of Hermite–Hadamard type inequalities for harmonically exponential type s–convex function, we need the following lemma:

Lemma 5.1 *[20] Let $\mathbb{Q}:[\eta_1,\eta_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a differentiable function and $\rho,\sigma\in[0,1]$.*

If $\mathbb{Q}' \in L[\eta_1,\eta_2]$, *then the following identity holds:*

$$\frac{\rho\mathbb{Q}(\eta_1)+\sigma\mathbb{Q}(\eta_2)}{2}+\frac{2-\rho-\sigma}{2}\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha$$
$$=\frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}\int_0^1[\frac{4(1-\rho-\omega)}{((1-\omega)\eta_2+(1+\omega)\eta_1)^2}\mathbb{Q}'(\frac{2\eta_1\eta_2}{(1-\omega)\eta_2+(1+\omega)\eta_1}) \tag{5.1}$$
$$+\frac{4(\sigma-\omega)}{(\omega\eta_1+(2-\omega)\eta_2)^2}\mathbb{Q}'(\frac{2\eta_1\eta_2}{t\eta_1+(2-\omega)\eta_2})]d\omega.$$

For simplicity, we denote

$$A_{\eta_1,\eta_2}=(1-\omega)\eta_2+(1+\omega)\eta_1 \quad and \quad B_{\eta_1,\eta_2}=\omega\eta_1+(2-\omega)\eta_2. \tag{5.2}$$

The following special functions will be used in sequel:

$$\Gamma(\eta)=\int_0^{+\infty}e^{-\omega}\omega^{\eta-1}d\omega,\eta>0;$$

$$\beta(\eta_1,\eta_2)=\int_0^1\omega^{\eta_1-1}(1-\omega)^{\eta_2-1}d\omega,\eta_1,\eta_2>0;$$

$$\beta(\eta_1,\eta_2)=\frac{\Gamma(\eta_1)\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)},\eta_1,\eta_2>0;$$

$${}_2F_1(\eta_1,\eta_2;\eta_3;\eta)=\frac{1}{\beta(\eta_2,\eta_3-\eta_2)}\int_0^1\omega^{\eta_2-1}(1-\omega)^{\eta_3-\eta_2-1}(1-\eta\omega)^{-\eta_1}d\omega,$$

where $\eta_3>\eta_2>0$ and $|\eta|<1$.

Theorem 5.2 *Let* $\mathbb{Q}:[\eta_1,\eta_2]\subseteq(0,+\infty)\to\mathbb{R}$ *be a differentiable function such that* $\mathbb{Q}'\in L[\eta_1,\eta_2]$ *and* $\rho,\sigma\in[0,1]$. *If the function* $|\mathbb{Q}'|^q$ *is harmonically exponential type* s*–convex, then for* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$|\frac{\rho\mathbb{Q}(\eta_1)+\sigma\mathbb{Q}(\eta_2)}{2}+\frac{2-\rho-\sigma}{2}\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$
$$\leq\eta_1\eta_2(\eta_2-\eta_1) \tag{5.3}$$
$$\times[\varphi_1^{\frac{1}{p}}(\Delta_1|\mathbb{Q}'(\eta_1)|^q+\Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}(\Delta_3|\mathbb{Q}'(\eta_1)|^q+\Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\varphi_1=\int_0^1|1-\rho-\omega|^p d\omega=\frac{(1-\rho)^{p+1}+\rho^{p+1}}{p+1},$$

$$\varphi_2=\int_0^1|\sigma-\omega|^p d\omega=\frac{(1-\sigma)^{p+1}+\sigma^{p+1}}{p+1},$$

$$\Delta_1=\frac{1}{2}\int_0^1\frac{1}{A_{\eta_1,\eta_2}^{2q}}(e^{s(1-\omega)}-1)d\omega,\Delta_2=\frac{1}{2}\int_0^1\frac{1}{A_{\eta_1,\eta_2}^{2q}}(e^{s(1+\omega)}-1)d\omega,$$

$$\Delta_3=\frac{1}{2}\int_0^1\frac{1}{B_{\eta_1,\eta_2}^{2q}}(e^{s(2-\omega)}-1)d\omega,\Delta_4=\frac{1}{2}\int_0^1\frac{1}{B_{\eta_1,\eta_2}^{2q}}(e^{s\omega}-1)d\omega,$$

and $A_{\eta_1,\eta_2},B_{\eta_1,\eta_2}$ are defined from (5.2).

Proof. From Lemma 5.1, Hölder's inequality, harmonically exponential type s–convexity of $|\mathbb{Q}'|^q$ and properties of modulus, we have

$$|\frac{\rho\mathbb{Q}(\eta_1)+\sigma\mathbb{Q}(\eta_2)}{2}+\frac{2-\rho-\sigma}{2}\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$

$$\leq \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}[\int_0^1 |\frac{4(1-\rho-\omega)}{((1-\omega)\eta_2+(1+\omega)\eta_1)^2}||\mathbb{Q}'(\frac{2\eta_1\eta_2}{(1-\omega)\eta_2+(1+\omega)\eta_1})|d\omega$$
$$+\int_0^1 |\frac{4(\sigma-\omega)}{(\omega\eta_1+(2-\omega)\eta_2)^2}||\mathbb{Q}'(\frac{2\eta_1\eta_2}{\omega\eta_1+(2-\omega)\eta_2})|d\omega]$$
$$\leq \eta_1\eta_2(\eta_2-\eta_1)\{(\int_0^1 |1-\rho-\omega|^p d\omega)^{\frac{1}{p}}$$
$$\times[\int_0^1 \frac{1}{A_{\eta_1,\eta_2}^{2q}}(\frac{1}{2}(e^{s(1-\omega)}-1)|\mathbb{Q}'(\eta_1)|^q+\frac{1}{2}(e^{s(1+\omega)}-1)|\mathbb{Q}'(\eta_2)|^q)d\omega]^{\frac{1}{q}}$$
$$+(\int_0^1 |\sigma-\omega|^p d\omega)^{\frac{1}{p}}$$
$$\times[\int_0^1 \frac{1}{B_{\eta_1,\eta_2}^{2q}}(\frac{1}{2n}(e^{s(2-\omega)}-1)|\mathbb{Q}'(\eta_1)|^q+\frac{1}{2n}(e^{s\omega}-1)|\mathbb{Q}'(\eta_2)|^q)d\omega]^{\frac{1}{q}}\}$$
$$=\eta_1\eta_2(\eta_2-\eta_1)$$
$$\times[\varphi_1^{\frac{1}{p}}(\Delta_1|\mathbb{Q}'(\eta_1)|^q+\Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}(\Delta_3|\mathbb{Q}'(\eta_1)|^q+\Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

which completes the proof.

Corollary 5.3 *Taking $\rho=\sigma$ in Theorem 5.2, then*

$$|\rho\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+(1-\rho)\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$
$$\leq \eta_1\eta_2(\eta_2-\eta_1)\varphi^{\frac{1}{p}}$$
$$\times[(\Delta_1|\mathbb{Q}'(\eta_1)|^q+\Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}+(\Delta_3|\mathbb{Q}'(\eta_1)|^q+\Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where $\varphi_1=\varphi_2=\varphi$.

Corollary 5.4 *Choosing $\rho=\sigma=0$ in Theorem 5.2, then*

$$|\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{2\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|\leq\frac{\eta_1\eta_2(\eta_2-\eta_1)}{\sqrt[p]{p+1}}$$
$$\times[(\Delta_1|\mathbb{Q}'(\eta_1)|^q+\Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}+(\Delta_3|\mathbb{Q}'(\eta_1)|^q+\Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}].$$

Corollary 5.5 *Choosing $\rho=\sigma=\frac{1}{2}$ in Theorem 5.2, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{2\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$
$$\leq\eta_1\eta_2(\eta_2-\eta_1)\sqrt[p]{\frac{4}{p+1}}$$
$$\times[(\Delta_1|\mathbb{Q}'(\eta_1)|^q+\Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}+(\Delta_3|\mathbb{Q}'(\eta_1)|^q+\Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}].$$

Corollary 5.6 *Taking $\rho=\sigma=\frac{1}{3}$ in Theorem 5.2, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+2\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{3\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$
$$\leq 3\eta_1\eta_2(\eta_2-\eta_1)\sqrt[p]{4(\frac{\left(\frac{2}{3}\right)^{p+1}+\left(\frac{1}{3}\right)^{p+1}}{p+1})}$$

$$\times [(\Delta_1|\mathbb{Q}'(\eta_1)|^q + \Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}} + (\Delta_3|\mathbb{Q}'(\eta_1)|^q + \Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}].$$

Corollary 5.7 *Taking* $\rho = \sigma = 1$ *in Theorem 5.2, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2} - \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha| \le \frac{\eta_1\eta_2(\eta_2-\eta_1)}{\sqrt[p]{p+1}}$$

$$\times [(\Delta_1|\mathbb{Q}'(\eta_1)|^q + \Delta_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}} + (\Delta_3|\mathbb{Q}'(\eta_1)|^q + \Delta_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}].$$

Theorem 5.8 *Let* $\mathbb{Q}: [\eta_1, \eta_2] \subseteq (0, +\infty) \to \mathbb{R}$ *be a differentiable function such that* $\mathbb{Q}' \in L[\eta_1, \eta_2]$ *and* $\rho, \sigma \in [0,1]$. *If the function* $|\mathbb{Q}'|^q$ *is harmonically exponential type* s–*convex, then for* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$|\frac{\rho\mathbb{Q}(\eta_1)+\sigma\mathbb{Q}(\eta_2)}{2} + \frac{2-\rho-\sigma}{2}\mathbb{Q}(\frac{\eta_1\eta_2}{\eta_1+\eta_2}) - \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$

$$\le \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4} \tag{5.4}$$

$$\times [\frac{4}{(\eta_1+\eta_2)^2}\left({}_2F_1\left(2p, 1; 2; \frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right)\right)^{\frac{1}{p}}(\Delta_5|\mathbb{Q}'(\eta_1)|^q + \Delta_6|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$

$$+\frac{1}{\eta_1^2}\left({}_2F_1\left(2p, 1; 2; \frac{\eta_1-\eta_2}{2\eta_1}\right)\right)^{\frac{1}{p}}(\Delta_7|\mathbb{Q}'(\eta_1)|^q + \Delta_8|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Delta_5 = \frac{1}{2}\int_0^1 |1-\rho-\sigma|^q(e^{s(1-\omega)}-1)d\omega,$$

$$\Delta_6 = \frac{1}{2}\int_0^1 |1-\rho-\sigma|^q(e^{s(1+\omega)}-1)d\omega,$$

$$\Delta_7 = \frac{1}{2}\int_0^1 |\sigma-\omega|^q(e^{s(2-\omega)}-1)d\omega, \Delta_8 = \frac{1}{2}\int_0^1 |\sigma-\omega|^q(e^{s\omega}-1)d\omega.$$

Proof. Applying Lemma 5.1, Hölder's inequality, harmonically exponential type s–convexity of $|\mathbb{Q}'|^q$ and properties of modulus, we have

$$|\frac{\rho\mathbb{Q}(\eta_1)+\sigma\mathbb{Q}(\eta_2)}{2} + \frac{2-\rho-\sigma}{2}\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}) - \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$

$$\le \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}[\int_0^1 |\frac{4(1-\rho-\omega)}{((1-\omega)\eta_2+(1+\omega)\eta_1)^2}||\mathbb{Q}'(\frac{2\eta_1\eta_2}{(1-\omega)\eta_2+(1+\omega)\eta_1})|d\omega$$

$$+\int_0^1 |\frac{4(\sigma-\omega)}{(\omega\eta_1+(2-\omega)\eta_2)^2}||\mathbb{Q}'(\frac{2\eta_1\eta_2}{\omega\eta_1+(2-\omega)\eta_2})|d\omega]$$

$$\le \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}\{4(\int_0^1 \frac{1}{A_{\eta_1,\eta_2}^{2p}}d\omega)^{\frac{1}{p}}$$

$$\times [\int_0^1 |1-\rho-\sigma|^q(\frac{1}{2n}\left(e^{s(1-\omega)}-1\right)|\mathbb{Q}'(\eta_1)|^q + \frac{1}{2n}\left(e^{s(1+\omega)}-1\right)|\mathbb{Q}'(\eta_2)|^q)d\omega]^{\frac{1}{q}}$$

$$+4(\int_0^1 \frac{1}{B_{\eta_1,\eta_2}^{2p}}d\omega)^{\frac{1}{p}}$$

$$\times [\int_0^1 |\sigma-\omega|^q(\frac{1}{2}\left(e^{s(2-\omega)}-1\right)|\mathbb{Q}'(\eta_1)|^q + \frac{1}{2}(e^{s\omega}-1)|\mathbb{Q}'(\eta_2)|^q)d\omega]^{\frac{1}{q}}\}$$

$$= \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}$$
$$\times [\frac{4}{(\eta_1+\eta_2)^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right))^{\frac{1}{p}}(\Delta_5|\mathbb{Q}'(\eta_1)|^q+\Delta_6|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\eta_1^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Delta_7|\mathbb{Q}'(\eta_1)|^q+\Delta_8|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

which completes the proof.

Corollary 5.9 *Taking $\rho=\sigma$ in Theorem 5.8, then*

$$|\rho\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+(1-\rho)\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|$$
$$\leq \frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}$$
$$\times [\frac{4}{(\eta_1+\eta_2)^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right))^{\frac{1}{p}}(\Lambda_1|\mathbb{Q}'(\eta_1)|^q+\Lambda_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\eta_1^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Lambda_3|\mathbb{Q}'(\eta_1)|^q+\Lambda_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Lambda_1=\frac{1}{2}\int_0^1 |1-2\rho|^q(e^{s(1-\omega)}-1)d\omega,$$

$$\Lambda_2=\frac{1}{2}\int_0^1 |1-2\rho|^q(e^{s(1+\omega)}-1)d\omega,$$

$$\Lambda_3=\frac{1}{2}\int_0^1 |\rho-\omega|^q(e^{s(2-\omega)}-1)d\omega,$$

$$\Lambda_4=\frac{1}{2}\int_0^1 |\rho-\omega|^q(e^{s\omega}-1)d\omega.$$

Corollary 5.10 *Taking $\rho=\sigma=0$ in Theorem 5.8, then*

$$|\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|\leq\frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}$$
$$\times [\frac{4}{(\eta_1+\eta_2)^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right))^{\frac{1}{p}}(\Lambda_5|\mathbb{Q}'(\eta_1)|^q+\Lambda_6|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\eta_1^2}(\quad {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Lambda_7|\mathbb{Q}'(\eta_1)|^q+\Lambda_8|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Lambda_5=\frac{1}{2}\int_0^1 (e^{s(1-\omega)}-1)d\omega,\Lambda_6=\frac{1}{2}\int_0^1 (e^{s(1+\omega)}-1)d\omega,$$

$$\Lambda_7=\frac{1}{2}\int_0^1 \omega^q(e^{s(2-\omega)}-1)d\omega,\Lambda_8=\frac{1}{2}\int_0^1 \omega^q(e^{s\omega}-1)d\omega.$$

Corollary 5.11 *Choosing $\rho=\sigma=\frac{1}{2}$ in Theorem 5.8, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{2\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|\leq\frac{\eta_1\eta_2(\eta_2-\eta_1)}{2}$$
$$\times[\frac{1}{\eta_1^2}(\ {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Lambda_9|\mathbb{Q}'(\eta_1)|^q+\Lambda_{10}|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Lambda_9=\frac{1}{2^{q+1}}\int_0^1|1-2\omega|^q(e^{s(2-\omega)}-1)d\omega,$$

$$\Lambda_{10}=\frac{1}{2^{q+1}}\int_0^1|1-2\omega|^q(e^{s\omega}-1)d\omega.$$

Corollary 5.12 *Choosing* $\rho=\sigma=\frac{1}{3}$ *in Theorem 5.8, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}+2\mathbb{Q}(\frac{2\eta_1\eta_2}{\eta_1+\eta_2})-\frac{3\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|\leq\frac{3\eta_1\eta_2(\eta_2-\eta_1)}{4}$$
$$\times[\frac{4}{(\eta_1+\eta_2)^2}(\ {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right))^{\frac{1}{p}}(\Omega_1|\mathbb{Q}'(\eta_1)|^q+\Omega_2|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\eta_1^2}(\ {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Omega_3|\mathbb{Q}'(\eta_1)|^q+\Omega_4|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Omega_1=\frac{1}{3^q2}\int_0^1(e^{s(1-\omega)}-1)d\omega,\Omega_2=\frac{1}{3^q2}\int_0^1(e^{s(1+\omega)}-1)d\omega,$$

$$\Omega_3=\frac{1}{3^q2}\int_0^1|1-3\omega|^q(e^{s(2-\omega)}-1)d\omega,\Omega_4=\frac{1}{3^q2}\int_0^1|1-3\omega|^q(e^{s\omega}-1)d\omega.$$

Corollary 5.13 *Taking* $\rho=\sigma=1$ *in Theorem 5.8, then*

$$|\frac{\mathbb{Q}(\eta_1)+\mathbb{Q}(\eta_2)}{2}-\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\mathbb{Q}(\alpha)}{\alpha^2}d\alpha|\leq\frac{\eta_1\eta_2(\eta_2-\eta_1)}{4}$$
$$\times[\frac{4}{(\eta_1+\eta_2)^2}(\ {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right))^{\frac{1}{p}}(\Omega_5|\mathbb{Q}'(\eta_1)|^q+\Omega_6|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\eta_1^2}(\ {}_2F_1\left(2p,1;2;\frac{\eta_1-\eta_2}{2\eta_1}\right))^{\frac{1}{p}}(\Omega_7|\mathbb{Q}'(\eta_1)|^q+\Omega_8|\mathbb{Q}'(\eta_2)|^q)^{\frac{1}{q}}],$$

where

$$\Omega_5=\frac{1}{2}\int_0^1(e^{s(1-\omega)}-1)d\omega,\Omega_6=\frac{1}{2}\int_0^1(e^{s(1+\omega)}-1)d\omega,$$

$$\Omega_7=\frac{1}{2}\int_0^1|1-\omega|^q(e^{s(2-\omega)}-1)d\omega,\Omega_8=\frac{1}{2}\int_0^1|1-\omega|^q(e^{s\omega}-1)d\omega.$$

6 Conclusion

In this paper, we have introduced and defined some algebraic properties of a new class of functions namely harmonically exponential type s –convex functions. A new version of Hermite–Hadamard type inequality and an integral identity for the differentiable functions are obtained. In recent years, many mathematicians put effort into the theory of inequalities to bring a new dimension to mathematical analysis. Due to widespread views and importance, this theory has become an attractive and absorbing field for scientists. We believe that our new class of functions will have very deep research in this fascinating field of inequalities and also in pure and applied sciences. The amazing concept of this paper can be extended to the co-ordinates along with fractional calculus.

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