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A Study on Connectedness in the Digital Topology Via Graph Theory

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Abstract:
In this paper we define at the two operators at Cartesian complex in digital topology based on graph theory and also investigate at the new classes of separation, connectedness and disconnectedness among the pixels with the topological axioms in digital plane. The related theorems are proved based on these concepts.

Keywords **— Cut point, pixels, interior operator, closure operator, separation, connectedness, disconnectedness.**

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I. Introduction

Digital topology is to study at the topological properties of digital, image arrays. The Cartesian complex have the collection of the pixel. In this case one can specify at the pixels on the simple closed curves which states that simple closed curves separate at the plane into two connected sets. When a pixel is extended to be such a set of pixels possess connectivity and is called a region. The use of digital topological ideas to explore various aspects of graph theory. A graph (resp., directed graph or digraph) (R.J. Wilson, 1996), G=(V(G), E(G)) consists of a vertex set V(G) and an edge set E(G) of unordered (resp., ordered) pairs of elements of V(G). To avoid ambiguities, we assume that the vertex and edge sets are disjoint. A subgraph (W.D. Wallis, 2007), of a graph G is a graph, each of whose vertices belong to V(G) and each of whose edges belong to E(G). A walk in which no vertex appears more than once is called a path. For other notions or notations in topology not defined here we follow closely (R. Englking, 1989; S. Willard, 1970).

II. Preliminaries

Definition 2.1[1]: A point x in X is called a cut point (respectively endpoint) if X-{x} has two (one) components. (In the literature our cut-point is usually called a strong cut-point, but here it turns out that these two notions coincide.) The parts of X-{x} are its components if there are two, and X-{x}, φ if there is only one.

Definition 2.2[5]: A nonempty set S is called a locally finite (LF) space if to each element e of S certain subsets of S are assigned as neighbourhoods of e and some of them are finite.

Definition 2.3 [5]: Axiom 1. For each space element e of the space S there are certain subsets containing e, which are neighbourhoods of e. The intersection of two neighbourhoods of eis again a neighbourhood of e. Since the space is locally finite, there exists the smallest neighbourhood of e that is the intersection of all neighbourhoods of e. Thus, each neighbourhood of e contains its smallest neighbourhood. We shall denote the smallest neighbourhood of e by SN(e).

Definition 2.4[5]: Axiom 2. There are space elements, which have in their SN more than one element.

Definition 2.5[5]: If $b \in SN(a)$ or $a \in SN(b)$, then the elements a and b are called incident to each other.

Definition 2.6[4]: A path is a sequence (p_i / $0 \leq i \leq n$), and p_i is adjacent to p_{i+1}. In another way Let Tbe a subset of the space S. In another way [5] a sequence ($a_1, a_2, \ldots, a_k$), $a_i, \in T$, i= 1, 2, . . ., k; in which each two subsequent elements are incident to each other, is called an incidence path in T from a_1 to a_k.

Definition 2.7 [4]: A set of pixels is said to be connected if there is a path between any two pixels.

Remark 2.8[5]: In another way A subset T of the space S is connected iff for any two elements of T there exists an incidence path containing these two elements, which completely lies in T

Definition 2.9 [5]: The topological boundary, also called the frontier, of a non-empty subset T of the space S is the set of all elements e of S, such that each neighbourhood of e contains elements of both T and its complement S–T. It is denoted by the frontier of $T \subseteq S$ by Fr (T, S).

Definition 2.9[5]: A subset $O \subset S$ is called open in S if it contains no elements of its frontier Fr (O, S). A subset $C \subset S$ is called closed in S if it contains all elements of Fr (C, S).
Definition 2.10[5]: The neighbourhood relation N is a binary relation in the set of the elements of the space S. The ordered pair (a, b) is in N iff $a \in SN(b)$. We also write aNb for (a, b) in N.

Definition 2.11[5]: A pair (a, b) of elements of the frontier Fr (T, S) of a subset $T \subset S$ are opponents of each other, if a belongs to SN(b), b belongs to SN(a), one of them belongs to T and the other one to its complement S – T.

Definition 2.12[5]: The smallest open subset of the ALF space S that contains the element $a \in S$ is called the smallest open neighborhood of a in S and is denoted by SON (a, S). It is denoted by SON (a, S) = SN(a).

Definition 2.13 [5]: The topology of a space S is defined if a collection of subsets of S is declared to be the collection of open subsets satisfying the following axioms:
Axiom C1. The entire set S and the empty subset $\varphi$are open.
Axiom C2. The union of any number of open subsets is open.
Axiom C3. The intersection of a finite number of open subsets is open.

Definition 2.14[3]: Let G=(V(G), E(G)) be a digraph, P(V(G)) its power set of all subgraphs of G and Cl_G: P(V(G)) →P(V(G)) is a mapping associating with each subgraph H=(V(H), E(H)) a subgraph $Cl_G(V(H)) \subseteq V(G)$ called the closure subgraph of H such that:$Cl_G(V(H)) = V(H) \cup \{v \in V(G) \backslash V(H); \overrightarrow{hv} \in E(G)$ for all $h \in V(H)\}$.The operation Cl_G is called graph closure operator and the pair (V(G), C_G) is called G-closure space, where $C_G(V(G))$ is the family of elements of Cl_G. The dual of the graph closure operator Cl_G is the graph interior operator Int_G: P(V(G)) →P(V(G)) defined by $Int_G(V(H)) = V(G) \backslash Cl_G(V(G) \backslash V(H))$ for all subgraph $H \subseteq G$. A family of elements of Int_G is called interior subgraph of H and denoted by $O_G(V(G))$. Clear that (V(G), O_G) is a topological space. Then the domain of Cl_G is equal to the domain of Int_G and also $Cl_G(V(H)) = V(G) \backslash Int_G(V(G) \backslash V(H))$. A subgraph H of G-closure space (V(G), C_G) is called closed subgraph if $Cl_G(V(H)) = V(H)$. It is called open subgraph if its complement is closed subgraph, i.e., $Cl_G(V(G) \backslash V(H)) = V(G) \backslash V(H)$, or equivalently $Int_G(V(H)) = V(H)$

Definition 2.15[3]: Let G=(V(G), E(G)) be a digraph and Cl_{Gm}:P(V(G)) →P(V(G)) an operator such that:
(a) It is called G_m-closure operator if $Cl_{Gm}(V(H)) = Cl_G(Cl_G(\ldots Cl_G(V(H))))$. m-times, for every subgraph $H \subseteq G$,
(b) it is called G_m-topological closure operator if $Cl_{Gm+1}(V(H)) = Cl_{Gm}(V(H))$ for all subgraph $H \subseteq G$. The space (V(G), C_{Gm}) is called G_m-closure space.

III. CONNECTEDNESS IN THE DIGITAL TOPOLOGY VIA GRAPH THEORY

DEFINITION:3.1

Let G=(V(G), E(G)) be a digraph and int_{Gm}:P(V(G)) →P(V(G)) an operator such that H=(V(H), E(H)) be subgraph then the **interior operator** in cartesian complex $int_{Gm}(V(H), V(G)) = \cup\{O_G(V(G))$ with the axioms C_1, C_2, C_3: (V(G), O_G) is an open subgraph in a pixel $int_G(V(G)) \subset V(H)\}$ where int_G: P(V(G)) →P(V(G)).

DEFINITION:3.2

Let G=(V(G), E(G)) be a digraph and Cl_G: P(V(G) →P(V(G)) an operator such that H=(V(H), E(H)) be subgraph then the **closure operator** in cartesian complex $Cl_{Gm}(V(H), V(G)) = \cap\{Cl_{Gm}(V(G))$ with the axioms C_1, C_2, C_3: (V(G), Cl_{Gm}) is a closed subgraph in a pixel $Cl_{Gm}(V(G)) \supset V(H)\}$ where Cl_{Gm}: P(V(G) →P(V(G)).

THEOREM:3.3

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)), B=(V(B), E(B)) are two subgraphs on G then, if (V(A), E(A)) □(V(B), E(B)), then int_{Gm}(V(A), V(G)) □int_{Gm}(V(B), V(G)).

Proof: Let (V(A), E(A)) and (V(B), E(B)) be a subgraph on G such that (V(A), E(A)) □(V(B), E(B)). Let $x \in int_{Gm}$(V(A), V(G)). Then there exists an open subgraphs U such that $x \in U$□(V(B), E(B)) and hence $x \in int_{Gm}$(V(B), V(G)). Hence int_{Gm}(V(A), V(G)) □int_{Gm}(V(B), V(G)).

THEOREM:3.4

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)), B=(V(B), E(B)) are two subgraphs on G then $int_{Gm}(V(A), V(G)) \cap (V(B), V(G)) = int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$.

Proof: We know that (V(A), E(A))$\cap$(V(B), E(B)) □(V(A), E(A)) and (V(A), E(A))$\cap$(V(B), E(B)) □(V(B), E(B)) we have by theorem 3.3, $int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$ □int_{Gm}(V(A), V(G)) and $int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$ □int_{Gm}(V(B), V(G)). This implies that $int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$ □$int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$ → (1). Again, $x \in int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$. Then, $x \in int_{Gm}$(V(A), V(G)) and $x \in int_{Gm}$(V(B), V(G))
Then there exists open subgraphs U=(V(U), E(G)) and W=(V(W), E(G)) such that $x \in U$□(V(A), V(G)) and $x \in W$□(V(B), V(G))∵$U \cap W$ is an open subgraph such that $x \in (U \cap W)$ □(VA), E(A)) ∩ (V(B), E(B)). Hence, $x \in int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$. Thus $x \in int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$ implies that $x \in int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$. Therefore, $int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$ □ $int_{Gm}(V(A), V(G)) \cap (V(B), V(G))$ → (2). from (1) and (2), if follows that $int_{Gm}(V(A), V(G)) \cap (V(B), V(G)) = int_{Gm}(V(A), V(G)) \cap int_{Gm}(V(B), V(G))$, Hence proved.

THEOREM:3.5

$int_{Gm}(V(O), V(G)) = \cup \{O_G(V(O))$: (V(O), O_G) is open subgraph, $V(O) \subset G\}$.

Proof:$a \in int_{Gm}(V(O), V(G)) \Leftrightarrow$ (V(O), E(G)) is a largest collection of open subgraphs of $a \Leftrightarrow$ there exists an open subgraph $(V(O), O_G)$ such that $a\in O_G$ □$G\Leftrightarrow a\in \cup\{V(O_G)$: $O_G(V(O))$: $(V(O), O_G)$ is open subgraph, $V(O)\subset G\}$. Hence, int_{Gm} $(V(O), V(G)) = \cup\{ O_G(V(O))$: $(V(O), O_G)$ is open, $O_G\subset G\}$.

THEOREM:3.6

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)) is a subgraph on G and let A be a subgraph on G then, $int_{Gm}(V(A), V(G))$ is an open subgraph.

Proof: Let p be any arbitrary point of $int_{Gm}(V(A), V(G))$. Then p is an interior graph of A. Hence by the definition, A is a largest collection of open subgraphs of p. Then there exists an open subgraph M such that p∈(V(M), E(M)) □(V(A), E(A)). Since (V(M), E(M)) is open subgraph, it is a largest collection of open subgraphs of each of its points and so A is also a largest collection of open subgraphs of each point of (V(M), E(M)). It follows that every point of (V(M), E(M)) is an open interior point of (V(A), E(A)) so that (V(M), E(M)) □$int_{Gm}(V(A), V(G))$. Thus, it is shown that to each p∈$int_{Gm}(V(A), V(G))$ there exists an open subgraph M such that p ∈ (V(M), E(M)) □ $int_{Gm}(V(A), V(G))$. Hence, $int_{Gm}(V(A), V(G))$ is a largest collection of open subgraphs of each of its points and consequently $int_{Gm}(V(A), V(G))$ is open subgraph.

THEOREM:3.7

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)) is a subgraph on G and let A be a subgraph on G then, $int_{Gm}(V(A), V(G))$ is the largest open graph contained in A.

Proof: Let M be a subgraph of A and let p∈(V(M), E(M)) so that p∈(V(M), E(M)) □(V(A), E(A)).Since, M is an open subgraph on G, A is a largest collection of open subgraphs of p and consequently p is an open interior point of A.Hence p∈int_{Gm} (V(A), V(G)).Thus, we have shown that p∈(V(M), E(M)) ⟹p∈int_{Gm} (V(A), V(G)) and so (V(M), E(M)) □int_{Gm} (V(A), V(G)) □A.Hence, int_{Gm} (V(A), V(G)) contains every open subgraph of A and it is therefore the largest open subgraph of A.

THEOREM:3.8

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)) is a subgraph on G and let A be a subgraph on G then A is an open if and only if $int_{Gm}(V(A), V(G))$ =A.

Proof: Let A=int_{Gm} (V(A), V(G)) by theorem 3.6, int_{Gm} (V(A), V(G)) is an open graph and therefore A is also an open graph. Conversely A is an open graph. Then, A is surely identical with the largest open subgraph of A. but int_{Gm} (V(A), V(G)) is the largest open subgraphs of A. Hence, A=int_{Gm} (V(A), V(G)).

DEFINITION:3.9

Let G=(V(G), E(G)) be a digraph.Two non-empty open subgraphs of A=(V(A), E(A)) and B=(V(B), E(B)) of G are said to be separated if and only if

[(V(A), E(A)) ⋂ (Cl_{Gm}((V(B), V(G))] ⋃[Cl_{Gm}((V(A), V(G))⋂V((B), E(B))] =φ.

THEOREM:3.10

If A=(V(A), E(A)) and B=(V(B), E(B)) are separated the subgraphs of a digraph on G and (V(C), E(C)) □(V(A), E(A)) and (V(D), E(D)) □ (V(B), E(B)) then C= (V(C), E(C)) and D=(V(D), E(D)) are also separated.

Proof: We are given that A ⋂ Cl_{Gm}(V(B), V(G)) = and Cl_{Gm}(V(A), V(G))⋂B = φ → (1) Also, (V(C), E(C)) □(V(A), E(A))⟹Cl_{Gm}(V(C), V(G)) □Cl_{Gm}(V(A), V(G)) And (V(D), E(D)) □(V(B), E(B))⟹Cl_{Gm}(V(D), V(G)) □Cl_{Gm}(V(B), V(G)) → (2) It follows from (1) and (2) that (V(C), E(C))⋂Cl_{Gm}(V(D), V(G)) = φ and Cl_{Gm}(V(C), V(G))⋂(V(D), E(D)) = φ. Hence C and D are separated.

DEFINITION:3.11

Let G=(V(G), E(G)) be a digraph. A subgraph A= (V(A), E(A)) which cannot be expressed as the union of two separated is said to be connected. In another way some discussion about the connected graph

1. If (V(A), E(A)) and (V(B), E(B)) are separated, then they are separated.
2. Every connected subgraph is a connected graph.
3. G is connected subgraph if and only if G is not the union of two non-empty disjoint open subgraphs if and only if
   G=(V(A), E(A)) ⋃ (V(B), E(B)); (V(A), E(A)) ⊂ (V(G), O_G); (V(B), E(B)) ⊂ (V(G), O_G); (V(A), E(A))≠ φ; (V(B), E(B)) ≠φ, implies that(V(A), O_G)⋂(V(B), O_G))≠ φ.

DEFINITION:3.12

Let G=(V(G), E(G)) be a digraph which is a union of two disjoint non-empty separated is called disconnected graph.

THEOREM:3.13

Let G=(V(G), E(G)) be a digraph which is connected if the only subgraph of G which are both open subgraph and closed subgraph are $\varphi$and G.

Proof: If X=(V(A), E(A))⋃(V(B), E(B)) with (V(A), O_G) and (V(B), O_G) are open subgraph and disjoint, then G-(V(A), O_G) = (V(B), O_G) and so (V(B), O_G) is the complement of open subgraph and hence is closed subgraph. Thus, (V(B), O_G) is disconnected. Similarly, (V(A), O_G) is disconnected. Conversely, if (V(A), O_G) is a non-empty proper open subgraph then (V(A), O_G) and G-(V(A), O_G) are disconnected of G.

THEOREM:3.14

Let G=(V(G), E(G)) be a digraph and subgraph A=(V(A), E(A)) on G is disconnected subgraph if and only if A is the union of two non-empty disjoint graph both open (closed)subgraph in A.

Proof: Let A be a subgraph of G and is disconnected graph if and only if there exist non-empty graph W=(V(W), E(W)) and H=(V(H), E(H)) both open (closed) subgraph in G such that V(W) ∩ V(A) $\neq \varphi$,V(H) ∩ V(A) $\neq \varphi$,(V(W) ∩ V(A)) ∩ (V(H) ∩ V(A)) $\neq \varphi$ and (V(W) ∩V(A)) ∩V(H) ∩V(A)) =A.

THEOREM:3.15

Let G=(V(G), E(G)) be a digraph and A=(V(A), E(A)) is a subgraph on G is disconnected subgraph if and only if there exist non-empty graph M=(V(M), O_G) and N=(V(N), O_G) both open (closed) subgraph in G such that V(M) ∩ V(A) $\neq \varphi$,V(N) ∩ V(A) $\neq \varphi$,V(A) ⊂ V(M) ∪ V(N) and V(M) ∩V(N)⊂V(G)-V(A).

Proof: By the above theorem, Y=(V(Y), E(Y)) is disconnected graph if and only if there exist non-empty graph V(M) and V(N) both open (closed) subgraph in G such that V(M) ∩ V(Y) $\neq \varphi$, V(N) ∩V(Y) $\neq \varphi$,(V(M) ∩V(Y)) ∩(V(N) ∩V(Y)) $\neq \varphi$ and (V(M) ∩ V(Y) ∪ (V(N) ∩ V(Y)) =Y. Now, (V(M) ∩ V(Y)) ∩ (V(N) ∩ V(Y)) $\neq \varphi$ ⇔(V(M) ∩V(N)) ∩V(Y) $\neq \varphi$ V(M) ∩V(N) ⊂ (G-Y) and (V(M) ∩V(Y)) ∪ (V(N) ∩V(Y)) =Y⇔(V(M) ∪ V(N)) ∩Y) =Y⇔Y⊂V(M) ∪ V(N).

THEOREM:3.16

Let G=(V(G), E(G)) be a digraph and subgraph M=(V(M), E(M)) is a connected subgraph of G such that V(M) ⊂V(A)∪V(B) where V(A) and V(B) are separated, Then either V(M) ⊂V(A) or V(M) ⊂ V(B).

Proof: Since V(A) and V(B) are separated, V(A) ∩ Cl_{Gm}(V(B), V(G)) $\neq \varphi$, Cl_{Gm}(V(A), V(G)) ∩ V(B) $\neq \varphi$. Now V(M) ⊂ V(A) ∪ V(B) ⟹ V(M) ∩ (V(A) ∪ V(B)) =(V(M) ∩V(A)) ∪ (V(M) ∩V(B)) →(1). We claim that at least one of the graph V(M) ∩V(A) and V(M) ∩V(B) are empty. For, if possible, suppose none of them is empty, That is, suppose that V(M) ∩V(A) $\neq \varphi$ and V(M) ∩V(B) $\neq \varphi$, Then, (V(M) ∩V(A)) ∩ (V(M) ∩V(B)) ⊂ (V(M) ∩V(A)) ∩ [Cl_{Gm}(V(M), V(G) ∩ Cl_{Gm}(V(B), V(G))] ⟹ (V(M) ∩ V(M)) ∩ [V(A) ∩ Cl_{Gm}(V(B), V(G))] ⟹ [V(M) ∩ Cl_{Gm}(V(M), V(G)] ∩ $\varphi \neq \varphi$ .Similarly,(V(M) ∩ V(A)) ∩ (V(M) ∩ V(B)) $\neq \varphi$ .Hence (V(M) ∩V(A)) and (V(M) ∩V(B)) are separated. Thus, M has been expressed as the union of two non-empty separated and consequently M is F disconnected. This is a contradiction. Hence at least one of the sets(V(M) ∩V(A)) and (V(M) ∩V(B)) is empty. if V(M) ∩V(A) $\neq \varphi$. Then (1) gives V(M)=(V(M) ∩ V(B)) which implies that V(M) ⊂ V(B).Similarly, if (V(M) ∩ V(B)) $\neq \varphi$, then V(M) ⊂V(A). Hence either V(M) ⊂V(A) or V(M) ⊂V(B).

COROLLARY:3.17

If M=V(M), E(M) is a connected subgraph of a graph G such that M⊂A∪B where A=(V(A), O_G), B=(V(B), O_G) are disjoint open (closed) subgraphs of G, then A and B are separated.

Proof: If A, B are open graph with A∩B$\neq \varphi$, Then(V(A), O_G)⊂G-(V(B), O_G) ⟹Cl_{Gm}(V(A), O_G)⊂Cl_{Gm}(G-(V(B), O_G) = G-(V(B), O_G) ⟹ Cl_{Gm}(V(A), O_G) ∩ B= φ . Similarly, A ∩ Cl_{Gm}(G-(V(B), O_G). Hence A, B is separated.

IV. CONCLUSIONS

In this paper we have defined at the two operators based on the wide area like graph theory and topology in the digital topology. These concepts will be applied and also extend in the other regions.

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