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Analysis of Differentials Equations and Get their Solutions using Sobolev Space

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Abstract: **Among many purposes of science, analyzing nature may be the most important and beautiful part. We all know that Differential Equations (DEs) are the mathematical expression of many natural phenomena. In modern science, analyzing tools like Calculus, Measure, sequence, series etc. have been used very frequently. Every analysis of DEs has only one goal which is to get the solutions of a DE. In fact, most of these DEs don't have exact solutions and many methods has been introduced to get some good solutions. Now-a-days, Functional Analysis plays an important role to analyze these methods. Some methods have solid foundation and flexibility. To make use all of these methods properly, we have to understand the nature of DEs and also realize the characteristics of the solutions. Without having any idea about solutions we don't think more, this is why we have started our study of analysis to take more and more benefits of these methods. This analysis will give us a solid platform to select best methods among others and will help us to find new more accurate methods.**

As usual, this analysis is covering the concepts of measure, which leads us to a more useful Integration concept. For finding the solutions of DEs some essential spaces have been introduced such as L^p spaces. The space concepts have been more specified into Sobolev Spaces.

Keywords: Sobolev Spaces, Normed linear space, trace operator, Minkowski's inequality, Morrey's inequality.

I Introduction

A. Normed Linear Spaces [1]

Definition-1: Let X be a linear space over the field of scalars 𝕂. A real-valued function $\|.\|$ on X is called a norm on X if it satisfies the following conditions:

(i) Non-negative; $\|x\| \in [0, \infty)$ for x ∈ X.

(ii) $\|x\| = 0$ if and only if x = 0 ∈ X.

(iii) Positive homogeneity; $\|\alpha x\| = |\alpha|\|x\|$ for x ∈ X and α ∈ 𝕂.

(iv) Triangle inequality; $\|x + y\| \le \|x\| + \|y\|$ for x, y ∈ X.

A linear space X with a norm $\|.\|$ defined on it is called a normed linear space and we write (X, $\|.\|$) for it.

Let $\|.\|$ be a norm on a linear space X. Then $|\|x\| - \|y\|| \le \|x - y\|$ for x, y ∈ X.

Definition-2: A sequence $(x_n : n \in N)$ in a normed linear space (X, $\|.\|$) is called a Cauchy sequence with respect to the norm $\|.\|$ if it is a Cauchy sequence with respect to the metric derived from the norm, that is, if for every ε ≥ 0 there exists N ∈ N such that $\|x_m - x_n\| < \varepsilon$ for m, n ≥ N.

Definition-3: A normed linear space (X, $\|.\|$) is complete with respect to the norm $\|.\|$ if every Cauchy sequence $(x_n : n \in N)$ with respect to the norm there exists x ∈ X such that

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

Definition-4: A normed linear space (X, $\|.\|$) is called a Banach space if (X, $\|.\|$) is complete with respect to the norm $\|.\|$.

Definition-5: Given a sequence $(x_n$: n ∈ N) in a normed linear space (X, $\|.\|$). Consider the sequence $(s_n$: n ∈ N) in X defined by $s_n = \sum_{k=1}^{n} x_k$ for n ∈ N. If the sequence $(x_n$: n ∈ N) converges in the norm, that is, if there exists s ∈

X such that $\lim_{n\to\infty}\|s_n - s\| = 0$, then we say that the series $\sum_{n\in \mathbb{N}} x_n$ converges in the norm to the sum s and write $\sum_{n\in \mathbb{N}} x_n$ = s. When no such s exists in X, we say that the series $\sum_{n\in \mathbb{N}} x_n$ diverges. We call (s_n : n ∈ N) the sequence of partial sums of the series $\sum_{n\in N} x_n$.

Theorem-1: Given a normed linear space (X,‖. ‖). The space X is complete with respect to the norm ‖. ‖ if and only if for every sequence (s_n : n ∈ N) in X such that $\sum_{n\in \mathbb{N}}\|x_n\| < \infty$, the series $\sum_{n\in \mathbb{N}} x_n$ converges in the norm.

Theorem-2: (Minkowski's inequality for $p\in[1,\infty)$): Given a measure space (X, 𝔄, μ). Let f and g be two extended complex-valued 𝔄-measurable functions on X such that $|f|,|g| < \infty$. Then for every $p\in[1,\infty)$, we have

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

B. The L^p Spaces for $p\in[1,\infty)$[2]

Given a measure space (X, 𝔄, μ). Let $p\in(0,\infty)$ and consider the linear space of functions $L^p(X,\mathfrak{A},\mu)$. The origin that is the identity of addition, of this linear space is the identically vanishing function 0 on X and for this function 0 we have $\|0\|_p = 0$. On the other hand if $f \in L^p(X,\mathfrak{A},\mu)$ and $\|f\|_p = 0$ then f(x) =0 for a.e. x ∈ X and f need not be the identically vanishing function on X. For this reason $\|.\|_p$ is not a norm on the linear space $L^p(X,\mathfrak{A},\mu)$.

Theorem-1: Given a measure space (X, 𝔄,μ). Consider a sequence ($f_n : n \in N$) and an element f in $L^p(X,\mathfrak{A},\mu)$ where p ∈ [0,∞). If the sequence converges to f in the L^p norm, that is, $\lim_{n\to\infty}\|f_n - f\|_p = 0$, then

(i) $\lim_{n\to\infty}\|f_n\|_p = \|f\|_p$

and hence $\lim_{n\to\infty}\int_X |f_n|^p d\mu = \int_X |f|^p d\mu$,

(ii) ($f_n : n \in N$) converges to f in measure μ on X,

(iii) There exists a subsequence ($f_{n_k} : n \in N$) such that $\lim_{k\to\infty} f_{n_k} = f$ a.e. on X.

Definition-1: Given a measure space (X, 𝔄, μ). Let f be an extended complex-valued 𝔄-measurable function on X. We define the essential supremum of f on X, $\|f\|_\infty$, as the infimum of the set of all essential bounds for f on X, that is,

$$\|f\|_\infty = \inf\{M\in[0,\infty) : \mu(x\in X : |f(x)| > M) = 0\}$$

If f has no essential bound on X, that is, if the set of all essential bounds of f on X is an empty set then we set $\|f\|_\infty$ = ∞. (This is consistent with the convention that the infimum of an empty subset of R, is equal to ∞. Recall that the smaller the set the greater the infimum over the set.)

Thus $\|f\|_\infty$ is defined for every extended complex-valued 𝔄-measurable function f on X; $\|f\|_\infty$ ∈ [c, ∞]; and $\|f\|_\infty = \infty$ if and only if f does not have an essential bound. We call $\|f\|_\infty$ the essential supremum of f. An alternate notation for $\|f\|_\infty$ is $\text{ess.}\sup_{x\in X}|f(x)|$.

C. Sobolev space [3]

Definition: For 1 ≤ p ≤ ∞ and let k be a nonnegative integer. The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions u :U →R such that for each multiindex α with |α| ≤ k, $D^\alpha(u)$ exists in the weak sense and belongs to $L^p(U)$. The natural number k is called the order of the Sobolev space $W^{k,p}(U)$.

D. Traces [4]

Next we discuss the possibility of assigning "boundary values" along ∂U to a function u∈ $W^{1,p}(U)$, assuming that ∂U is C^1. Now if u ∈ C(U), then clearly u has values on ∂U in the usual sense. The problem is that a typical function u ∈ $W^{1,p}(U)$ is not in general continuous and, even worse, is only defined a.e. in U. Since ∂U has n-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "u restricted to ∂U". The notion of a trace operator resolves this problem. For this section we take 1 ≤ p< ∞.

Theorem-1: (Trace theorem): Assume U is bounded and ∂U is C^1. Then here exists a bounded linear operator

$$T: W^{1,p}(U) \to L^p(\partial U)$$

Such that

(i) $Tu = u|_{\partial U}$ if u ∈ $W^{1,p}(U) \cap C(\bar{U})$, and

(ii) $\|Tu\|_{L^p(\partial U)} \le C\|u\|_{W^{1,p}(U)}$

For each u ∈ $W^{1,p}(U)$, with the constant C depending only on p and U. We call Tu the trace of u and ∂U.

Theorem-2: (Estimates for $W^{1,p}$, 1 ≤ p < n): Let U be a bounded, open subset of R^n, and suppose ∂U is C^1. Assume 1 ≤ p < n, and u ∈ $W^{1,p}$(U). Then u ∈ L^{p^*}(U), with the estimate

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)} \quad (4.28)$$

The constant C depends only on p, n, and U.

Theorem-3 [5]*: (Estimates for $W_0^{1,p}$, 1 ≤ p < n):* Let U be a bounded, open subset of R^n, and suppose u ∈ $W_0^{1,p}$(U) for some 1 ≤ p < n. Then we have the estimate

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)}$$

for each q ∈ [1, p^*], the constant C depending only on p, q, n and U.

T is estimate is sometimes called Poincare's inequality.

Theorem-4: (Morrey's inequality: Assume n < p ≤ ∞. Then there exists a constant C, depending only on p and n, such that

$$\|u\|_{C^{0,\gamma}(R^n)} \leq C\|u\|_{W^{1,p}(R^n)}$$

for all $u \in C^1(R^n)$, where $\gamma = 1-{}^{n}/_{p}$.

II Methodology

A. If $u \in W^{k,p}(U)$, we define its norm to be

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \leq p \leq \infty) \\ \sum_{|\alpha|\leq k} ess\sup_U |D^\alpha u| & (p=\infty) \end{cases}$$

(i) Let $\{u_m\}_{m=1}^{\infty}, u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$, written

$$u_m \to u \text{ in } W^{k,p}(U),$$

Provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0$$

(ii) We write

$$u_m \to u \text{ in } W^{k,p}_{loc}(U),$$

to mean

$$u_m \to u \text{ in } W^{k,p}(V),$$

for each $V \subset\subset U$.

B. We denote by $W_0^{k,p}(U)$, the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Thus $u \in W_0^{k,p}(U)$ if and only if there exist a functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. We interpret $W_0^{k,p}(U)$ as comprising those functions $u \in W^{k,p}(U)$ such that

$$"D^\alpha u = 0 \text{ on } \partial U" \text{ for all}$$

$|\alpha| \leq k-1$

C. ***General Sobolev inequalities:*** Let U be a bounded open subset of R^n, with a C^1 boundary. Assume

$$u \in W^{k,p}(U).$$

i. If

$$k < \frac{n}{p},$$

Then

$$u \in L^q(U),$$

Where,

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

We □*ave* in addition the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

The constant C depending only on $k, p, n \,\&\, U$.

ii. If

$$k > \frac{n}{p},$$

Then, $u \in C^{k-\left|\frac{n}{p}\right|-1,\gamma}(\overline{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, if\ \frac{n}{p}\ is\ not\ an\ integer \\ any\ positive\ number < 1, if\ \frac{n}{p}\ is\ an\ integer. \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left|\frac{n}{p}\right|-1,\gamma}(U)} \leq C\|u\|_{W^{k,p}(U)},$$

The constant C depending only on $k, p, n, \gamma \,\&\, U$.

iii. Results

A. *Sobolev space as function space:* For each k = 1,2,……. And 1 ≤ p≤ ∞, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Proof: Let us first of all check that $\|u\|_{W^{k,p'}(U)}$ is a norm. Clearly

$$\|\lambda u\|_{W^{k,p'}(U)} = |\lambda|\|u\|_{W^{k,p'}(U)},$$

and

$$\|u\|_{W^{k,p'}(U)} = 0 \qquad \text{if and only if } u = 0$$

Next assume $u, v \in W^{k,p}(U)$ .Then if 1 ≤ p < ∞, Minkowski's inequality implies

$$\|u+v\|_{W^{k,p'}(U)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)}\right)^{1/p}$$

$$\leq \left(\sum_{|\alpha|\leq k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)}\right)^p\right)^{1/p}$$

$$\leq \sum_{|\alpha|\leq k} \left(\|D^\alpha u\|^p_{L^p(U)}\right)^{1/p} + \left(\|D^\alpha v\|^p_{L^p(U)}\right)^{1/p}$$

$$= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}$$

It remains to show that $W^{k,p}(U)$ is complete. So assume $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(U)$.Then for each $|\alpha|\le k, \{D^{\alpha}u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(U)$. Since $L^p(U)$ is complete, there exist function $u_\alpha\in L^p(U)$ such that

$$D^{\alpha}u_m \to u_\alpha \qquad \text{in } L^p(U)$$

For each $|\alpha|\le k$ in particular,

$$u_m \to u_{(0,0,......,0)} \qquad \text{in } L^p(U).$$

We now claim $u\in W^{k,p}(U), D^{\alpha}(u)=u_\alpha \ (|\alpha|\le k)$ (1)

To verify this assertion, fix $\phi\in C_c^{\infty}(U)$ and then

$$\int_U u D^{\alpha}\phi dx = \lim_{m\to\infty}\int_U u_m D^{\alpha}\phi dx$$

$$= \lim_{m\to\infty}(-1)^{|\alpha|}\int_U D^{\alpha}u_m \phi dx$$

$$= (-1)^{|\alpha|}\int_U u_\alpha \phi dx$$

Thus (1) is valid. Since therefore $D^{\alpha}u_m \to D^{\alpha}u$ in $L^p(U)$ for all $|\alpha|\le k$, we see that $u_m\to u$ in $W^{k,p}(U)$, as required.

B. *Local approximation by smooth functions:* Assume $u\in W^{k,p}(U)$ for some $1\le p<\infty$ and set

$$u^{\varepsilon} = \eta_{\varepsilon} * u \qquad \text{In } U_{\varepsilon}$$

Then

i. $u^{\varepsilon}\in C_c^{\infty}(U_{\varepsilon})$ for each $\varepsilon>0$,

and

ii. $u^{\varepsilon}\in u$ in $W_{loc}^{k,p}(U)$, as $\varepsilon\to 0$.

Proof: Clearly (i) is true. We next claim that if $|\alpha|\le k$, then

$$D^{\alpha}(u^{\varepsilon}) = \eta_{\varepsilon} * D^{\alpha}u \ \text{ in } U_{\varepsilon}. \tag{2}$$

That is, the ordinary α^{th} - partial derivative of the smooth function u^{ϵ} is the ε-mollification of the α^{th} - weak partial derivative of u to confirm this, we compute for $x\in U_{\varepsilon}$

$$D^{\alpha}u^{\varepsilon}(x) = D^{\alpha}\int_U \eta_{\varepsilon}(x-y)u(y)dy$$

$$= \int_U D_x^{\alpha}\eta_{\varepsilon}(x-y)u(y)dy$$

$$= (-1)^{|\alpha|}\int_U D_y^{\alpha}\eta_{\varepsilon}(x-y)u(y)dy$$

Now for fixed $x\in U_{\varepsilon}$ the function $\phi(y)=\eta_{\varepsilon}(x-y)$ belongs to C_c^{∞}.Consequently the definition of the α^{th}-weak partial derivative implies:

$$\int_U D_y^{\alpha}\eta_{\varepsilon}(x-y)u(y)dy = (-1)^{|\alpha|}\int_U \eta_{\varepsilon}(x-y)D^{\alpha}u(y)dy$$

Thus

$$D^{\alpha}u^{\varepsilon}(x) = (-1)^{|\alpha|+|\alpha|}\int_U \eta_{\varepsilon}(x-y)D^{\alpha}u(y)dy$$

$$= [\eta_{\varepsilon} * D^{\alpha}u](x)$$

This establishes (2).

Now choose an open set $V\subset\subset U$ .In view of (4.11), $D^{\alpha}(u^{\varepsilon})\to D^{\alpha}u$ in $L^p(V)$ as $\varepsilon\to 0$ for each $|\alpha|\le k$. Consequently

$$\|u^{\varepsilon}-u\|_{W^{k,p}(V)}^{p} = \sum_{|\alpha|\le k}\|D^{\alpha}u^{\varepsilon}-D^{\alpha}u\|_{L^p(V)}^{p}\to 0$$

as $\varepsilon\to 0$. This proves assertion (ii).

C. *Global approximation by smooth functions:* Assume U is bounded, and suppose as well that $u\in W^{k,p}(U)$ for some $1\le p<\infty$. Then there exist functions $u_m\in C^{\infty}(U) \cap W^{k,p}(U)$ such that

$$u_m\to u \text{ in } W^{k,p}(U)$$

Proof: We have $U=\bigcup_{i=1}^{\infty}U_i$, where

$$U_i = \{x\in U \mid dist(x,\partial U) > 1/i\} \qquad (i=1,2,.......).$$

Write $V_i = U_{i+3} - \overline{U_{i+1}}$.

Choose also any open set $V_0\subset\subset U$ so that $U=\bigcup_{i=0}^{\infty}V_i$. Now let $\{\zeta_i\}_{i=0}^{\infty}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^{\infty}$; that is, suppose

$$\begin{cases} 0 \le \zeta_i \le 1, & \zeta_i \in C_c^\infty(V_i) \\ \sum_{i=0}^{\infty} \zeta_i = 1 & on U. \end{cases}$$

Next, choose any function $u \in W^{k,p}(U)$. According to theorem 4.1, $\zeta_i u \in W^{k,p}(U)$ and $spt(\zeta_i u) \subset V_i$.

Fix $\delta > 0$. Choose then $\varepsilon_i > 0$ so small that $u^i = \eta \varepsilon_i * (\zeta_i u)$ satisfies

$$\begin{cases} \left\| u^i - \zeta_i u \right\|_{W^{k,p}(U)} \le \frac{\delta}{2^{i+1}} & (i = 0,1,......) \\ spt\, u^i \subset W_i & (i = 1,......), \end{cases}$$

For $W_i = U_{i+4} - \overline{U}_i \supset V_i \quad (i = 1,......)$.

Write $v = \sum_{i=0}^{\infty} u^i$. This function belongs to $C^\infty(U)$, since for each open set $V \subset\subset U$ there are at most finitely many nonzero terms in the sum. Since $u = \sum_{i=0}^{\infty} \zeta_i u$, we have for each $V \subset\subset U$

$$\left\| v - u \right\|_{W^{k,p}(V)} \le \sum_{i=0}^{\infty} \left\| u^i - \zeta_i u \right\|_{W^{k,p}(U)}$$

$$\le \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

Take the supremum over sets $V \subset\subset U$, to conclude $\left\| v - u \right\|_{W^{k,p}(U)} \le \delta.$

If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u$ (Leibniz' formula) (3)

Where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Now prove (3) by induction on $|\alpha|$. Suppose first $|\alpha| = 1$. Choose any $\phi \in C_c^\infty(U)$. Then

$$\int_U \xi u D^\alpha \phi dx = \int_U u D^\alpha(\xi\phi) - u(D^\alpha \xi)\phi dx$$

$$= -\int_U (\xi D^\alpha(u) + u D^\alpha(\xi))\phi dx$$

Thus $D^\alpha(\zeta u) = \zeta D^\alpha u + u D^\alpha \zeta$, as required.

Next assume $l < k$ and formula (3) is valid for all $|\alpha| \le l$ and all functions ζ. Choose a multi-index α with $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. Then for ϕ as above,

$$\int_U \xi u D^\alpha \phi dx = \int_U \xi u D^\beta (D^\gamma \phi) dx$$

$$= (-1)^{|\beta|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\beta-\sigma} u D^\gamma \phi dx$$ (by the induction assumption)

$$= (-1)^{|\beta|+|\gamma|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \xi D^{\beta-\sigma} u) \phi dx$$ b (by the induction assumption sign)

$$= (-1)^{|\alpha|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} [D^\sigma \xi D^{\alpha-\rho} u + D^\rho \xi D^{\alpha-\sigma} u] \phi dx$$

(Where $\rho = \sigma + \gamma$)

$$= (-1)^{|\alpha|} \int_U [\sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u] \phi dx$$

Since $\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}$.

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