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Subspace of the Vector Space

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Abstract:

The paper presents the result of the research on Subspace of the vector space presented. In the research work we used foreign reliable sources and materials.Linear algebra is the study of linear maps on finite-dimensional vector spaces and subspace. In this paper we will define subspace and will discuss their properties. In many applications in linear algebra, vector spaces occur as subspaces of larger spaces. For instance, as we know that the solution set of a homogeneous system of linear equations in variables is a subspace. A nonempty subset of a vector space is a subspace when it is a vector space (with the same operations defined in the original vector space). Subspace and their ancillary structures provide the common language of linear algebra, and, as such, are an essential prerequisite for understanding contemporary applied (and pure) mathematics.

Keywords **—Vector space, subspace, vector, set and subset.**

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I. INTRODUCTION

Let us begin with the definition of vector space, which is general structure for the subspace.

Definition 1. 1 Let F be a field, whose elements are referred to as scalars. A vector space vectors over F is a nonempty set V, whose elements are referred to as vectors, together with two operations. The first operation, called addition and denoted by $(+)$, assigns to each pair (u, v) of vectors in V a vector $u + v$ in V. The second operation, called scalar multiplication and denoted by juxtaposition, assigns to each pair $(r, u) \in F \times V$ a vector ru in V. Furthermore, the following properties must be satisfied:

1) (Associativity of addition) For all vectors $u, v, w \in V$,

$$u + (v + w) = (u + v) + w$$

2) (Commutativity of addition) For all vectors $u, v \in V$

$$u + v = v + u$$

3) (Existence of a zero) There is a vector $0 \in V$ with the property that

$$0 + u = u + 0 = u$$

for all vectors $u \in V$.

4) (Existence of additive inverses) For each vector $u \in V$, there is a vector in V, denoted by $-u$, with the property that

$$u + (-u) = (-u) + u = 0$$

5) (Properties of scalar multiplication) For all scalars $a, b \in F$ and for all

Vectors $u, v \in V$.

$$a(u + v) = au + av$$
$$(a + b)u = au + bu$$
$$(ab)u = a(bu)$$
$$1u = u$$

Note that the first four properties in the definition of vector space can be summarized by saying that is an abelian group under addition.

Most algebraic structures contain substructures, and vector spaces are no exception, hence the substructure of the vector space is a subspace which is the main part of the research paper.

Definition2. A subspace of a vector space V is a subset W of V that is a vector space in its own right under the operations obtained by restricting the operations of V to W. We use the notation $W \leq V$ to indicate that W is a subspace of V and $W < V$ to indicate that W is a proper subspace of V, that is, $W \leq V$ but $W \neq V$. The zero subspace of V is $\{0\}$[5, p. 36, 37].

For more clarification, if W is a subset of V, then to check that W is a subspace of V we need only check that W satisfies the following:

additive identity $0 \in W$

closed under addition $w, v \in W$ implies $w + v \in W$;

closed under scalar multiplication $a \in F$ and $w \in W$ implies $aw \in W$.

The first condition insures that the additive identity of V is in W. second condition insures that addition makes sense on W. The third condition insures that scalar multiplication makes sense $x_1$on W. To show that W is a vector space, the other parts of the definition of a vector space do not need to be checked because they are automatically satisfied. For example, the associative and commutative properties of addition automatically hold on W because they hold on the larger space V. As another example, if the third condition above holds and $w \in W$, then $-w$ (which equals $(-1)w$) is also in W, and hence every element of W has an additive inverse in W. The three conditions above usually enable us to determine quickly whether a given subset of V is a subspace of V[4, p. 13].

Remark. 1. First, note that a vector subspace of a vector space must contain the zero vector. Indeed, say W is a vector subspace of V . Since W is nonempty, there is $v \in W$, but then $0 = 0v \in W$. Thus if a subset W of a vector space V does not contain the zero vector, then this subset W has no chance of being a vector subspace of V.

2. Next, a key observation is that if W is a subspace of V, then W becomes itself an F-vector space, by restricting the operations in V to W [6, p. 114].

Example 1. Let $V = R^4$, which is a vector space over R. We claim that $W = \{(a, b, 2a - b + 3c, c): a, b, c \in R\}$ is a subspace of V. Letting $a = b = c = 0$,we see that $(0, 0, 0, 0) \in W$. To check closure under addition, take $a_i, b_i, c_i \in R$. Then

$$(a_1, b_1, 2a_1 - b_1 + 3c_1, c_1) + (a_2, b_2, 2a_2 - b_2 + 3c_2, c_2)$$
$$= (a_1 + a_2, b_1 + b_2, 2(a_1 + a_2) - (b_1 + b_2) + 3(c_1 + c_2), c_1 + c_2) \in W.$$

Similarly, if , $a \in R$, then

$$a(a_1, b_1, 2a_1 - b_1 + 3c_1, c_1) = (aa_1, a\, b_1, 2aa_1 - ab_1 + 3ac_1, ac_1) \in W.$$

Thus, we have closure under scalar multiplication, and the claim is proved. [1, p. 209]

Example 2. A Subspace of$M_{2,2}$. Let W be the set of all 2×2symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

Solution. Recall that a square matrix is symmetric when it is equal to its own transpose. Because $M_{2,2}$ is a vector space, we only need to show that W a subset of $M_{2,2}$. Begin by observing that W is nonempty. W is closed under addition because for matrices A_1 and A_2 in W, $A_1 = {A_1}^T$and $A_2 = {A_2}^T$, which implies that

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2.$$

So, if A_1 and A_2 are symmetric matrices of order 2, then so is $A_1 + A_2$. Similarly, W is closed under scalar multiplication because $A = A^T$ implies that $(cA)^T = cA^T = cA$. If A is a symmetric matrix of order 2, then so is cA.

The result of Example 1 can be generalized. That is, for any positive integer n, the set of symmetric matrices of order n is a subspace of the vector space $M_{2,2}$ with the standard operations. [3, p. 163].

2. Properties of The Subspace.

1. Intersection of a family of subspaces is a subspace.

Proof. Let $\{W_\alpha / \alpha \in \Lambda\}$ be a family of subspaces of a vector space V over F. Then $0 \in W_\alpha$ for all α, and so 0 belongs to the intersection of the family. Thus, the intersection of the given family is nonempty. Let $x, y \in \bigcap_{\alpha\in\Lambda} W_\alpha$, and a, b ∈ F. Then $x, y \in W_\alpha$ for all α. Since each W_α is a subspace, $ax + by \in W_\alpha$ for all α. Hence $ax + by$ belongs to the intersection. This shows that the intersection of the family is a subspace.

2. Union of subspaces need not be a subspace. Indeed, the union $W_1 \cup W_2$ of two subspaces is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. If $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$ a subspace. Similarly, if $W_2 \subseteq W_1$, then also the union is a subspace. Conversely, suppose that $W_1 \cup W_2$. Is a subspace and W_1 is not a subset of W_2 .Then there is an element $x \in W_1$ which is not in W_2 . Let $y \in W_2$. Then, since $W_1 \cup W_2$ is a subspace , $x + y \in W_1 \cup W_2$. Now $x + y$ does not belong to W_2,for otherwise $x = (x + y) - y$ will be in W_2, a contradiction to the supposition. Hence $x + y \in W_1$. Since $x \in W_1$ and W_1 is subspace, $y = -x + (x + y)$belongs to W_1 . This shows that $W_2 \subseteq W_1$.

3. Let W_1 and W_2 be subspaces of a vector space V over a field F. Then $W_1 + W_2 = \{x + y / x \in W_1, y \in W_2$ is also a subspace (called the sum of W_1 and W_2) which is the smallest subspace containing $W_1 \cup W_2$.

Proof. Since $0 \in W_2$, $x \in W_1$ implies that $x = x + 0 \in W_1 + W_2$. Thus, $W_1 \subseteq W_1 + W_2$ .Similarly, $W_2 \subseteq W_1 + W_2$. Also, if L is a subspace containing $W_1 \cup W_2$, then $x + y \in L$ for all $x \in W_1$, and $y \in W_2$. Therefore, it is sufficient to show that $W_1 + W_2$. Is a subspace. Clearly, $W_1 + W_2 \neq \emptyset$. Let $x + y$ and $u + v$ belong to $W_1 + W_2$, where $x, u \in W_1$,and $y, v \in W_2$ .Since W_1 and W_2 are subspaces $\alpha x + \beta u \in W_1$, and $\alpha y + \beta v \in W_2$. But, then $\alpha(x + y) + \beta(u + v) = (\alpha x + \beta u) + (\alpha y + \beta v)$ belongs to $W_1 + W_2$. [2, p. 12].

We can extend it to W_n . That if $W_1, W_2, \dots, W_n$ are subspaces of a vector space V ,then $W_1 + W_2 + \cdots + W_n$ is a subspace of V .

Now we define a very important operation on subspaces of an F-vector space:

Definition 3. Let $W_1, W_2, \dots, W_n$ be subspaces of a vector space V . Their sum $W_1 + W_2 + \cdots + W_n$ is the subset of V consisting of all vectors $w_1 + w_2 + \cdots + w_n$ with $w_1 \in W_1, \dots, w_n \in W_n$.

Example 3. Prove that $W_1 + W_2 + \cdots + W_n$ is the smallest subspace of V containing all subspaces $W_1, \dots, W_n$.

Solution. It is clear that $W_1 + \cdots + W_n$ contains $W_1, W_2, \dots, W_n$ since each vector w_i of W_i can be written as $0 + 0 + \cdots + w_i + 0 + \cdots + 0$ and $0 \in W_1 \cap \dots \cap W_n$. We need to prove that if W. is any subspace of V which contains each of the subspaces $W_1, \dots, W_n$, then W contains $W_1 + W_2 + \cdots + W_n$ .Take any vector v of $W_1 + \cdots + W_n$. By definition, we can write $v = w_1 + w_2 +$

$\cdots + w_n$ for some vectors $w_i \in W_i$. Since W contains $W_1, \ldots, W_n$, it contains each of the vectors $w_1, \ldots, w_n$. And since W is a subspace of V, it must contain their sum, which is v. We proved that any element of $W_1 + \cdots + W_n$ belongs to W, thus
$W_1 + \cdots + W_n \subset W$ and the result follows.
We now introduce a second crucial notion, that of direct sum of subspaces:

Definition 4. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. We say that $W_1, W_2, \ldots, W_n$ are in direct sum position if the equality
$w_1 + w_2 + \cdots + w_n = 0$
with $w_1 \in W_1, \ldots, w_n \in W_n$ forces $w_1 = w_2 = \cdots = w_n = 0$.

Finally, we make another key definition:
Definition 5. a) We say that a vector space V is the direct sum of its subspaces $W_1, W_2, \ldots, W_n$ and write

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

if $W_1, W_2, \ldots, W_n$ are in direct sum position and $V = W_1 + W_2 + \cdots + W_n$.
b) If V_1, V_2 are subspaces of a vector space V, we say that V_2 is a complement (or complementary subspace) of V_1 if $V_1 \oplus V_2 = V$.

By the previous results, $V = W_1 \oplus \ldots \oplus W_n$ if and only if every vector $v \in V$ can be uniquely written as a sum $w_1 + w_2 + \cdots + w_n$ with $w_i \in W_i$ for all i. Hence, if V_1, V_2 are subspaces of V, then V_2 is a complement of V_1 if and only if every vector $v \in V$ can be uniquely expressed as $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$.

Example 4. 1. The vector space $V = R^2$ is the direct sum of its subspaces $V_1 = \{(x, 0)/x \in R\}$and $V_2 = \{(0, y)/y \in R\}$. Indeed, any $(x, y) \in R^2$ can be uniquely in the form $(a, 0) + (0, b)$, via $a = x$ and $b = y$.

2. Let $V = M_n(R)$ be the vector space of $n \times n$ matrices with real entries. If V_1 and V_2 are the subspaces of symmetric, respectively skew-symmetric matrices, then $V = V_1 \oplus V_2$. Indeed, any matrix $A \in V$ can be uniquely written as the sum of a symmetric matrix and a skew-matrix matrix: the only way to have $A = B + C$ with B symmetric and C skew-symmetric is via $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$[6, p.117, 118, 119].

Conclusion

In this paper we formalized the concept of vector subspace of a given vector space and then clarified some of its basic properties. The key concepts of subspace of the vector space, will appear, not only in linear systems of algebraic equations and the geometry of n-dimensional Euclidean space, but also in the analysis of linear di□erential equations, linear boundary value problems, Fourier analysis, signal processing, numerical methods, and many, many other fields. Therefore, first we defined vector space which is general structure for subspace. Then we clarified the main part of the paper subspace with its properties, and for more clarification we brought some useful examples.

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