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Performance Of A Multistage Rocket : Burnout Altitude

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ABSTRACT

A. Miele[6] analyzed the performance of a multistage rocket to the extent of obtaining burnout velocity. Further investigation into formulating the burnout altitude of a vertically flown multistage rocket in vacuum is not done and is not published. Present author,SN Maitra published three/four papers on performance of a multistage rocket on different aspects. Let us first introduce definitions[1] pertinent to single-stage and multistage rockets.

INTRODUCTION

Denoting by m_i the initial mass, m_f the final mass, m_p the propellant mass, m_s the structural mass (tanks, engines, pipes etc),m_* the payload mass and observing the definitions[1]

$$m_i = m_p + m_f, \quad m_f = m_s + m_* \qquad (1)$$

The dimensionless ratios[1]

$$\pi = \frac{m_*}{m_i}, \zeta = \frac{m_p}{m_i}, \epsilon = \frac{mp}{m_s + m_p} \qquad (2)$$

which are respectively called payload ratio, propellant mass ratio and structural factor. By use of equations (1) and (2) several relationships can be derived depicting dimensionless groups characterizing a single- stage rocket;vide table 1

$\frac{m_f}{m_i}$	1-ζ	$\in + \pi(1 - \in)$
$\frac{m_*}{m_i}$	$\frac{1 - \in - \zeta}{1 - \in}$	π
$\frac{m_p}{m_i}$	ζ	$(1 - \in)(1 - \pi)$
$\frac{m_s}{m_i}$	$\frac{\in \zeta}{1 - \in}$	$\in (1 - \pi)$

In order to analyze performance of a multistage rockets, let n denote the total number of stages, k the generic stage, m_{pk} and m_{sk} the propellant and structural masses of the k th stage and m_{ik} and m_{fk} the initial and final masses of the kth stage. Then for all subsequent stages we have for each stage

$$m_{ik} = m_{pk} + m_{fk} \qquad m_{fk} = m_{sk} + m_{i(k+1)} \qquad (1,2,3 \dots n) \qquad (3)$$

$$\text{where } \pi_k = \frac{m_{i(k+1)}}{m_{ik}} \qquad \zeta_k = \frac{m_{pk}}{m_{ik}} \qquad \epsilon_k = \frac{m_{sk}}{m_{sk}+m_{pk}} \qquad (4)$$

indicate payload ratio, propellant mass and structural factor of the kth stage and obviously these definitions are formally identical with those of a single-stage rocket and can be established in table2

$\frac{m_{pk}}{m_{ik}}$	ζ_k	$(1-\epsilon_k)(1-\pi_k)$
$\frac{m_{fk}}{m_{ik}}$	$1-\zeta_k$	$\epsilon_k + (1-\epsilon_k)\pi_k$
$\frac{m_{sk}}{m_{ik}}$	$\frac{\epsilon_k \zeta_k}{1-\epsilon_k}$	$\epsilon_k (1-\pi_k)$
$\frac{m_{i(k+1)}}{m_{ik}}$	$\frac{1-\epsilon_k - \zeta_k}{1-\epsilon_k}$	π_k
$m_{ik} = m_{i1}\pi_1\pi_2\pi_3 \dots \dots \dots \dots \pi_{k-1}$		

$$\text{Again,} \quad m_0 = m_{i1},\ m_* = m_{i(n+1)},\ m_{p0} = \sum_{k=1}^{n} m_{pk} \qquad (5)$$

respectively denote the over-all mass, payload mass, propellant mass of the rocket. Hence the over-all payload ratio and over-all propellant mass ratio can be defined as

$$\pi_0 = \frac{\pi_*}{m_0} \qquad \zeta_0 = \frac{m_{p0}}{m_0} \qquad (6)$$

Simple application of algebraic technique to the above yields relationships between over-all ratios and partial ratios

$$\pi_0 = \Pi_{k=1}^{n}\pi_k \text{ and } \zeta_0 = \sum_{k=1}^{n} \zeta_k \, \Pi_{j=1}^{k-1}\pi_j \qquad (6.1)$$

BURNOUT VELOCITY AND AITITUDE WITH CONSTANT THRUST OF THE MULTISTAGE ROCKET

Let us first consider performance of the multistage rocket in the kth stage with the following parameters:

V_k = final/burnout velocity in the kth stage

m_{fk} = final mass in the kth stage

m_{ik} = initial mass in the kth stage

β_k =propellant mass flow ie rate of propellant consumption in the kth stage

V_{Ek} = equivalent exit velocity in the kth stage

x_k = altitude attained in the kth stage from the level of projection of the multistage rocket

t_k=time of completion of the kth stage ie time to acquire velocity V_k from the initial velocity V_{k-1} which is the final velocity in the(k-1)th stage

V_{k-1} = final/ burnout velocity in the (k-1) th stage =initial velocity at time t=0 in the kth stage; g= the acceleration due to gravity

Then as done by A.Miele[6], solving the governing equation of motion for completion of the kth stage,

$$V_k = V_{k-1} - V_{Ek} \log \frac{m_{fk}}{m_{ik}} - gt_k \qquad (7)$$

At the same time with the above initial conditions, the velocity at height x at any time t during propellant burning in the kth stage ie $0<t<t_k$ is given by

$$\frac{dx}{dt} = V = V_{k-1} - V_{Ek} \log \frac{m}{m_{ik}} - gt \qquad (8)$$

So as to find the height attained by the multistage rocket due to operation of the kth stage we use in (8) the equation

$$\frac{dm}{dt} + \beta_k = 0 \qquad (9)$$

where m is the mass and x the height at any time t during the kth stage, so that

$$\frac{dx}{dm} = V_{k-1} \frac{dt}{dm} + \frac{V_{Ek}}{\beta_k} \log \frac{m}{m_{ik}} - gt\frac{dt}{dm} \qquad (10)$$

Utilizing the initial conditions as indicated one can solve (10) to get the part of the height gained due to the kth stage

$$h_k = x_k - x_{k-1} = V_{k-1}(t_k - t_{k-1}) + \frac{V_{Ek}}{\beta_k}\{\left(\log\frac{m_{fk}}{m_{ik}}\right)m_{fk} - (m_{fk} - m_{ik})\}$$

$$-\frac{1}{2}g(t_k - t_{k-1})^2 \quad (11)$$

For the kth stage, time elapses from (k-1) th stage to kth stage is $(t_k - t_{k-1})$.

If m_{pk} be the propellant mass consumed in the kth stage mentioned in the above nomenclature, we have from derivations[4]

$$V_{k-1} = \sum_{j=1}^{k-1}\{V_{Ej}\log\frac{m_{ij}}{m_{fj}} - g(\frac{m_{pj}}{\beta_j})\} \text{ and } t_k - t_{k-1} = \frac{m_{pk}}{\beta_k} \quad (12)$$

so that the preceding equation (11) becomes

$$h_k = x_k - x_{k-1}$$

$$=\sum_{j=1}^{k-1}\{V_{Ej}\log\frac{m_{ij}}{m_{fj}} - g(\frac{m_{pj}}{\beta_j})\}\frac{m_{pk}}{\beta_k} + \frac{V_{Ek}}{\beta_k}\left(\log\frac{m_{fk}}{m_{ik}}\right)m_{fk} - (m_{fk} - m_{ik})\} - \frac{1}{2}g(\frac{m_{pk}}{\beta_k})^2 \quad (13)$$

Now summing up for k=1,2,3……. n, where $x_0 = 0$, we get height H attained after exhaustion of n stages and at the same time we use table2:

$$H=\sum_{k=1}^{n}[\sum_{j=1}^{k-1}\{V_{Ej}\log\frac{m_{ij}}{m_{fj}} - g(\frac{m_{pj}}{\beta_j})\}\frac{m_{pk}}{\beta_k} + \frac{V_{Ek}}{\beta_k}\{\left(\log\frac{m_{fk}}{m_{ik}}\right)m_{fk} - (m_{fk} - m_{ik})\} -$$

$$\frac{1}{2}g(\frac{m_{pk}}{\beta_k})^2] \quad (14)$$

$$H=x_n = \sum_{k=1}^{n}[\sum_{j=1}^{k-1}\{V_{Ej}\log\frac{1}{\epsilon_j+(1-\epsilon_j)\pi_j} - (\frac{gm_{pj}}{\beta_j})\}(1-\epsilon_k)(1-\pi_k)\frac{m_{ik}}{\beta_k}]$$

$$-\sum_{k=1}^{n}[\frac{V_{Ek}}{\beta_k}[(\log\frac{1}{\epsilon_k+(1-\epsilon_k)\pi_k})\{\epsilon_k + (1-\epsilon_k)\pi_k\}m_{ik} - (1-\epsilon_k)(1-\pi_k)m_{ik}] -$$

$$\frac{1}{2}g\sum_{k=1}^{n}\{\frac{(1-\epsilon_k)(1-\pi_k)m_{ik}}{\beta_k}\}^2 \quad (14.1)$$

$$x_n=\sum_{k=1}^{n}[\sum_{j=1}^{k-1}\{V_{Ej}\log\frac{1}{\epsilon_j+(1-\epsilon_j)\pi_j} - (\frac{g}{\beta_j})\}(1-\epsilon_k)(1-$$

$$\pi_k)\pi_1\pi_2\pi_3\ldots\pi_{j-1}m_{i1})\}.\frac{(1-\epsilon_k)(1-\pi_k)}{\beta_k}\pi_1\pi_2\pi_3\ldots m_{i1}\pi_{k-1}m_0]$$

$$-\sum_{k=1}^{n}[\frac{V_{Ek}}{\beta_k}[(\log\frac{1}{\epsilon_k+(1-\epsilon_k)\pi_k})\{\epsilon_k + (1-\epsilon_k)\pi_k\}\pi_1\pi_2\pi_3\ldots m_{i1}\pi_{k-1}m_0$$

$$-(1-\epsilon_k)(1-\pi_k)\pi_1\pi_2\pi_3\ldots m_{i1}\pi_{k-1}m_0]$$

$$-\frac{1}{2g}\sum_{k=1}^{n}\left\{\frac{(1-\epsilon_k)(1-\pi_k)\pi_1\pi_2\pi_3\ldots\pi_{k-1}m_0}{\beta_k}\right\}^2$$

$$=H_1+H_2+H_3 \text{ (say)} \qquad m_0 = m_{i1} \qquad (15)$$

If there are equal payload ratios,structural factors,exit velocty and propellant mass flows for all stages ie $\pi_k = \pi$, $\beta_k = \beta$, $\epsilon_k = \epsilon$ and $V_{Ek} = V_E$ (16) where for the same constant thrust β is constant, then (15) gives

$$H_1=\sum_{k=1}^{n}\left[\left\{(k-1)V_E\log\frac{1}{\epsilon+(1-\epsilon)\pi} - g\sum_{j=1}^{k-1}\frac{(1-\epsilon)(1-\pi)m_0\pi^{j-1}}{\beta}\right\}\frac{(1-\epsilon)(1-\pi)m_0}{\beta}\pi^{k-1}\right]$$

$$=\frac{m_0}{\beta}V_E\left\{\log\frac{1}{\epsilon+(1-\epsilon)\pi}\right\}(1-\epsilon)(1-\pi)\sum_{k=1}^{n}[\{(k-1)\pi^{k-1} - g\left\{\frac{(1-\epsilon)(1-\pi)m_0}{\beta}\right\}^2\sum_{k=1}^{n}\left[\frac{1-\pi^{k-1}}{(1-\pi)}\pi^{k-1}\right]$$

(k=1,2 ,3…….n)

Now to evaluate the two above summations, we write

$$S_n = \sum_{k=1}^{n}(k-1)\pi^{k-1} = \sum_{k=1}^{n}k\pi^{k-1} - \sum_{k=1}^{n}\pi^{k-1} = S_n^1 - (1-\pi^n)/(1-\pi) \quad \text{where}$$

$$S_n^1 = \sum_{k=1}^{n}k\pi^{k-1} = 1+2\pi+3\pi^2+4\pi^3+\ldots\ldots n\pi^{n-1} = \frac{d}{d\pi}\left(\sum_{k=1}^{n}\pi^k\right)$$

$$=\frac{d}{d\pi}\{((1-\pi^{n+1})/(1-\pi)\}$$

$$(1-\pi)S_n^1 = S_n^1 - \{\pi+2\pi^2+3\pi^3+\cdots(n-2)\pi^{n-2}+(n-1)\pi^{n-1}\} - n\pi^n$$

$$=1+\pi+\pi^2+\pi^3+\cdots\pi^{n-1} - n\pi^n = (1-\pi^n)/(1-\pi) - n\pi^n$$

$$S_n^1 = \frac{1-\pi^n}{(1-\pi)^2} - \frac{n\pi^n}{1-\pi}$$

$$\text{Hence } S_n = \frac{1-\pi^n}{(1-\pi)^2} - \frac{n\pi^n}{1-\pi} - \frac{1-\pi^n}{1-\pi} = \frac{\pi(1-\pi^n)-n\pi^n(1-\pi)}{(1-\pi)^2} \qquad (16.1)$$

$$R_n = \sum_{k=1}^{n}(1-\pi^{k-1})\pi^{k-1} = \sum_{k=1}^{n}\pi^{k-1} - \sum_{k=1}^{n}(\pi^2)^{k-1} = \frac{1-\pi^n}{1-\pi} - \frac{1-\pi^{2n}}{1-\pi^2} \qquad (17)$$

Using (16) and (17) in the above expression for H_1, one gets

$$H_1=\frac{m_0}{\beta}V_E\left\{\log\frac{1}{\epsilon+(1-\epsilon)\pi}\right\}\frac{1-\epsilon}{1-\pi}\{\pi(1-\pi^n) - n\pi^n(1-\pi)\}$$

$$-\frac{gm_0^2(1-\epsilon)^2(1-\pi^n)(1-\pi^{n-1})\pi}{\beta^2(1+\pi)} \qquad (18)$$

$$-H_2 = \sum_{k=1}^{n}\frac{m_0V_E}{\beta_k}\left[\left\{\log\frac{1}{\epsilon_k+(1-\epsilon_k)\pi_k}\right\}\{\epsilon_k+(1-\epsilon_k)\} - (1-\epsilon_k)(1-\pi_k)\right]\pi^{k-1}$$

$=\sum_{k=1}^{n}\frac{m_0V_E}{\beta}\left[\left\{\log\frac{1}{\epsilon+(1-\epsilon)\pi}\right\}\{\epsilon+(1-\epsilon)\}-(1-\epsilon)(1-\pi)\right]\pi^{k-1}$

$H_2=\frac{m_0V_E}{\beta}\left[-\left\{\log\frac{1}{\epsilon+(1-\epsilon)\pi}\right\}\{\epsilon+(1-\epsilon)\}+(1-\epsilon)(1-\pi)\right]\frac{1-\pi^n}{1-\pi}$ (19)

$-H_3=\frac{m_0^2g}{2\beta^2}\{(1-\epsilon)(1-\pi)\}^2\sum_{k=1}^{n}\pi^{2(k-1)}=\frac{m_0^2g}{2\beta^2}\{(1-\epsilon)(1-\pi)\}^2\frac{1-\pi^{2n}}{1-\pi^2}$

$=\frac{m_0^2g}{2\beta^2}(1-\epsilon)^2(1-\pi)\frac{1-\pi^{2n}}{1+\pi}$ (20)

Then using equations (18), (19) and (20),the total time of vertical flight by the multistage rocket on reaching the burnout velocity is given by

$T=\sum_{K=1}^{n}\frac{m_{pk}}{\beta_k} =\sum_{k=1}^{n}\frac{(1-\epsilon_k)(1-\pi_k)m_{i1\pi_1\pi_2.....\pi_{k-1}}}{\beta_k} =\sum_{k=1}^{n}\frac{(1-\epsilon)(1-\pi)\pi^{k-1}}{\beta}m_{i1}$

$=\frac{(1-\epsilon)(1-\pi)(1-\pi^n)m_{i1}}{\beta(1-\pi)}=\frac{(1-\epsilon)(1-\pi^n)m_0}{\beta}$ (21)

However the greatest height attained by the multistage rocket is

$H=H_1+H_2+H_3+(V_b^2/2g)$ (22)

where $H_1+H_2+H_3$ gives the burout height whereas V_b is the burnout velocity obtained by A Miele[4] after jettisoning n stages:

$V_b=nV_E\log\frac{1}{\epsilon+(1-\epsilon)\pi}-\frac{(1-\epsilon)(1-\pi^n)m_0g}{\beta}$ (23)

PERFORMANCE WITH GIVEN INITIAL THRUST– TO –WEIGHT RATIO

If the multistage rocket is flown vertically upwards with given initial thrust-to-weight ratio in each stage, in view of the above analysis ,the burnout velocity[4] and burnout altitude can be determined in the following way:

$\beta_k=\frac{gm_{ik}\tau_{ik}}{V_{Ek}}, \qquad t_k-t_{k-1}=\frac{m_{pk}}{\beta_k}$ (24)

replacing which in the foregoing expressions we have

$V_{fk}=V_{ik}+V_{Ek}\log\frac{m_{ik}}{m_{fk}}-g(t_{fk}-t_{ik})$ where $t_{fk}=t_k$ and $t_{ik}=t_{k-1}$ (25)

For $t_{k-1} < t < t_k$, $V_{ik} = V_{f(k-1)} = V_{k-1}$

$$V=\frac{dx}{dt} = V_{k-1} - \frac{V_{Ek}^2}{g m_{ik}\tau_{ik}}\left(\log\frac{m}{m_{ik}}\right)\frac{dm}{dt} - g(t - t_{k-1}) \quad (26)$$

$$\beta_k = -(\frac{dm}{dt})_k \quad (26.1)$$

integrating which subject to the initial conditions as above, we get

$$h_k = x_k - x_{k-1} = V_{k-1}(t_{fk} - t_{ik}) - \frac{V_{Ek}^2}{g\tau_{ik}}\left\{\frac{m_{fk}}{m_{ik}}\log\frac{m_{fk}}{m_{ik}} + \frac{m_{ik} - m_{fk}}{m_{ik}}\right\}$$

$$-\frac{g}{2}(t_k - t_{k-1})^2$$

Summing up for k=1,2,3 ...n and substituting for $(t_{fk} - t_{ik})$ and V_{k-1} in the above equation, the burnout altitude(vide table2) H is given by

$$H=x_n - x_0=\sum_{k=1}^{n}[\{\sum_{j=1}^{k-1} V_{Ej}\log\frac{1}{\epsilon_j+(1-\epsilon_j)\pi_j} - \sum_{j=1}^{k-1}\frac{m_{pj}V_{EJ}}{m_{ij}\tau_{ij}}\}$$

$$x(1-\epsilon_k)(1-\pi_k)\frac{m_{ik}V_{Ek}}{m_{ik}\tau_{ik}g}] - \sum_{k=1}^{n}\frac{V_{EK}^2}{g\tau_{IK}}\left\{\frac{m_{fk}}{m_{ik}}\log\frac{m_{fk}}{m_{ik}} + \frac{m_{ik}-m_{fk}}{m_{ik}}\right\} -$$

$$\frac{g}{2}\sum_{k=1}^{n}(\frac{m_{pk}V_{Ek}}{m_{ik}\tau_{ik}g})^2 \quad (27)$$

$$gH =\sum_{k=1}^{n}[\sum_{j=1}^{k-1} V_{Ej}\{\log\frac{1}{\epsilon_j+(1-\epsilon_j)\pi_j} - \frac{(1-\epsilon_j)(1-\pi_j)}{\tau_{ij}}\}(1-\epsilon_k)(1-\pi_k)\frac{V_{Ek}}{\tau_k}]$$

$$-\sum_{k=1}^{n}\frac{V_{Ek}^2}{\tau_{ik}}\left\{\log\frac{1}{\epsilon_k + (1-\epsilon_k)\pi_k}\right\}\{\epsilon_k + (1-\epsilon_k)\} - (1-\epsilon_k)(1-\pi_k)\}$$

$$-\frac{1}{2}\sum_{k=1}^{n}\frac{V_{Ek}^2}{\tau_{ik}^2}\{(1-\epsilon_k)(1-\pi_k)\}^2 \quad =h_1+h_2+h_3 \quad (28)$$

For identical values of π_k, ϵ_k, τ_{ik}, V_{Ek} as done in earlier relationship (28), are obtained

$$gh_1\tau_i = \sum_{k=1}^{n} V_E(k-1)\left\{\log\frac{1}{\epsilon_k+(1-\epsilon_k)\pi_k} - \frac{(1-\epsilon)(1-\pi)}{\tau_i}\right\}(1-\epsilon)(1-\pi)V_E$$

$$=\frac{n(n-1)}{2}\{\log\frac{1}{\epsilon+(1-\epsilon)\pi} - \frac{(1-\epsilon)(1-\pi)}{\tau_i}\}(1-\epsilon)(1-\pi)V_E^2 \quad (29)$$

$$g h_2 \tau_i = -\sum_{k=1}^{n}\{\epsilon_k + (1-\epsilon_k)\pi_k\}\} \log \frac{1}{\epsilon_k + (1-\epsilon_k)\pi_k} - (1-\epsilon_k)(1-\pi_k)] V_{Ek}^2$$

$$= n V_E^2 \, [-\{\epsilon + (1-\epsilon)\pi_k\} \log \frac{1}{\epsilon + (1-\epsilon)\pi} + (1-\epsilon)(1-\pi)] \qquad (30)$$

$$2 g h_3 \tau_i^2 = -\sum_{k=1}^{n}\{V_{Ek}(1-\epsilon_k)(1-\pi_k)\}^2 = -\, n\{(1-\epsilon)(1-\pi)V_E\}^2 \qquad (31)$$

In consideration of (28) to (31), over- all burnout height is obtained by use of table1:

$$H = \frac{n V_E^2}{g \tau_i}[-\frac{n\zeta^2}{2\tau_i} + \zeta + (1 - \frac{n+1}{2}\zeta)\log(1-\zeta)] \qquad (32)$$

which is also obtained as exercise by A.Miele[4] other parameters remaining the same. With given final thrust-to-weight ratio τ_f in each stage to determine the over-all burnout height we need to replace τ_i by $\tau_f\{\epsilon + (1-\epsilon)\pi\}$ in equation (32). The maximum height can be determined as earlier, vide equation(22). In case of rocket design[1] with identical dimensionless groups characterizing parameters in each stage,we have

$$\pi_k = \pi, \quad \zeta = \zeta_k, \quad \epsilon_k = \epsilon \quad (k=1,2,3, \ldots\ldots n) \qquad (33)$$

such that the overall payload and propellant mass ratios become

$$\pi_0 = \pi^n \quad, ie, \pi = (\pi_0)^{\frac{1}{n}} \qquad (34)$$

$$\zeta_0 = \zeta \sum_{k=1}^{n} \pi^{k-1} = \zeta \frac{1-\pi^n}{1-\pi} = \zeta \frac{1-\pi_0}{1-(\pi_0)^{\frac{1}{n}}}$$ which by use of table2 gives

$$\zeta_0 = (1-\epsilon)(1-\pi_0) \qquad (35)$$

BURNOUT PERFOMANCE WITH INFINITE NUMBER OF STAGES

This is hypothetically evaluated from academic point of view.

The case with constant thrust throughout the powered flight: :In the light of the above if $n \to \infty$, (21)because of (34) yields

$$\text{Burning time} = T_\infty = \frac{(1-\epsilon)(1-\pi_0)}{\beta} m_0 \qquad (36)$$

By use of (23), the limiting burnout velocity $V_{*\infty}$ is given by

$$V_{*\infty} = Lim_{n\to\infty}[nV_E \log\frac{1}{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}} - \frac{(1-\epsilon)(1-\pi_0)}{\beta} m_0\, g]$$

.

$$=Lim_{(\frac{1}{n})} \to 0\{\frac{V_E \log\frac{1}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}}{\frac{1}{n}}\}(\frac{0}{0}) - \frac{(1-\epsilon)(1-\pi_0)}{\beta} m_0\, g$$

$$=-Lim_{(\frac{1}{n})} \to 0 \left[V_E(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0\right] - \frac{(1-\epsilon)(1-\pi_0)}{\beta} m_0\, g$$

$$=(1-\epsilon)\left\{V_E\log\frac{1}{\pi_0} - \frac{(1-\pi_0)}{\beta} m_0\, g\right\} \qquad (37)$$

In order to find the limiting value of the burnout height H_∞ as $n \to \infty$, in view of equations (18) to (20) and (34) we find the following limits

$$\mathrm{Lim}_{n\to\infty}[-\log\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\}\frac{\pi_0{}^{\frac{1}{n}}(1-\pi_0)}{1-(\pi_0)^{\frac{1}{n}}}\left(\frac{0}{0}\right) + \mathrm{Lim}_{n\to\infty}\frac{\pi_0\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}}{\frac{1}{n}}](1-\epsilon)$$

$$=Lim_{\frac{1}{n}\to 0}\ [\frac{-\pi_0^{\frac{1}{n}}(\log\pi_0)(1-\epsilon)\pi_0^{\frac{1}{n}}(1-\pi_0)}{\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}(-\log\pi_0)\pi_0^{\frac{1}{n}}} + \frac{\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}(1-\pi_0)\pi_0^{\frac{1}{n}}\log\pi_0}{\pi_0^{\frac{1}{n}}\log\pi_0}$$

$$+\frac{(1-\epsilon)\pi_0}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}](1-\epsilon) = \{(1-\pi_0)+\pi_0\}(1-\epsilon)^2 = (1-\epsilon)^2 \qquad (38)$$

$$\mathrm{Lim}_{n\to\infty}\frac{(1-\pi_0)(\pi_0^{\frac{1}{n}}-\pi_0)}{1+\pi_0^{\frac{1}{n}}} = (1-\pi_0)^2 \qquad (39)$$

so that $H_{1\infty} = \frac{m_0}{\beta}[V_E - \frac{m_0 g}{\beta}(1-\pi_0)^2](1-\epsilon)^2$

$$\mathrm{Lim}_{n\to\infty}\left\{\frac{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}{1-\pi_0^{\frac{1}{n}}}\right\}\{\log\frac{1}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}\}(1-\pi_0)(\frac{0}{0})$$

$$= \mathrm{Lim}_{n\to\infty}[\frac{-(1-\epsilon)\pi_0^{\frac{1}{n}}(\log\pi_0)\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}}{-\pi_0^{\frac{1}{n}}\log\pi_0}(1-\pi_0)$$

$$= \mathrm{Lim}_{n\to\infty}(1-\pi^{n})(1-\epsilon) = (1-\pi_0)(1-\epsilon) \qquad (40)$$

so that $H_{2\infty} = (1-\pi_0)(1-\epsilon) - (1-\pi_0)(1-\epsilon) = 0 \qquad (41)$

$$\mathrm{Lim}_{n\to\infty}\frac{\left(1-\pi_0^{\frac{1}{n}}\right)(1-\pi_0^{2})}{1+\pi_0^{\frac{1}{n}}} = 0 \text{ implies } H_{3\infty} = 0 \qquad (42)$$

so that $H_{\infty} = H_{1\infty} = \frac{m_0}{\beta}[V_E - \frac{m_0 g}{\beta}(1-\pi_0)^2](1-\epsilon)^2 \qquad (43)$

which gives the burnout altitude with infinite number of stages.

The case with the same initial thrust-to- weight/ final thrust-to- weight ratio in all the stages of the multistage rocket:

Though this is given in exercise[1] we prove herein the result with the help of table1, equation (32) and(33) along with (34):

$$H_{1i\infty} = \mathrm{Lim}_{n\to\infty}\frac{V_E^2}{g\tau_i}[\frac{-\left\{n(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\right\}^2}{2\tau_i} + n(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)$$

$$+n\{1-\frac{n+1}{2}(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\}\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}] =$$

$$\frac{V_E^2}{g\tau_i}(L_1 + L_2 + L_3) \qquad (44)$$

$$L_1 = -\frac{1}{2\tau_i}\mathrm{Lim}_{n\to\infty}[\frac{\left\{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\right\}^2}{\frac{1}{n}}\left(\frac{0}{0}\right) = -\frac{(1-\epsilon)^2}{2\tau_i}\mathrm{Lim}_{n\to\infty}\left(\pi_0^{\frac{1}{n}}\log\pi_0\right)^2$$

$$= -\frac{(1-\epsilon)^2}{2\tau_i}(\log\pi_0)^2 \qquad (45)$$

$$L_2 = (1-\epsilon)\mathrm{Lim}_{n\to\infty}\frac{\left(1-\pi_0^{\frac{1}{n}}\right)}{\frac{1}{n}}\left(\frac{0}{0}\right) \quad = -(1-\epsilon)\mathrm{Lim}_{n\to\infty}\,\pi_0^{\frac{1}{n}}\log\pi_0$$

$$= -(1-\epsilon)\log\pi_0 \qquad (46)$$

$$L_3 = \text{Lim}_{n\to\infty} n\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}\left(\frac{0}{0}\right) - \frac{1}{2}\text{Lim}_{n\to\infty}\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}}{\frac{1}{n(n+1)}}\left(\frac{0}{0}\right)$$

$$= \text{Lim}_{\frac{1}{n}\to 0}\frac{(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}} - \frac{1}{2}\text{Lim}_{\frac{1}{n}\to 0}\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}}{\frac{1}{n}-(1-\frac{1}{1+\frac{1}{n}})}\left(\frac{0}{0}\right)$$

$$= (1-\epsilon)\log\pi_0 - \frac{(1-\epsilon)}{2}\text{Lim}_{\frac{1}{n}\to 0}[(-\pi_0^{\frac{1}{n}}\log\pi_0)\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}$$

$$+\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)\pi_0^{\frac{1}{n}}\log\pi_0}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}]\frac{1}{1-\left(\frac{1}{1+\frac{1}{n}}\right)^2}\left(\frac{0}{0}\right)$$

$$= (1-\epsilon)\log\pi_0 - \frac{(1-\epsilon)}{2}\text{Lim}_{\frac{1}{n}\to 0}[-(1-\epsilon)\left(\pi_0^{\frac{1}{n}}\log\pi_0\right)^2\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}^{-1}$$

$$+\left(1-\pi_0^{\frac{1}{n}}\right)\frac{d}{d\frac{1}{n}}\left\{\frac{(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}\right\} - \left(\pi_0^{\frac{1}{n}}\log\pi_0\right)^2\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}$$

$$-(1-\epsilon)\left(\pi_0^{\frac{1}{n}}\log\pi_0\right)^2\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\}^{-1}]\frac{1}{\frac{2}{(1+\frac{1}{n})^3}}$$

$$L_3 = (1-\epsilon)\log\pi_0 + \frac{(1-\epsilon)^2}{2}(\log\pi_0)^2 \qquad (47)$$

Using the limiting values (45) to (47) in equation (44) is obtained the overall the burnout altitude

$$H_\infty = H_{1i\infty} = \frac{V_E^2}{2g\tau_i}(1-\frac{1}{\tau_i})(1-\epsilon)^2(\log\pi_0)^2 \qquad (48)$$

In case of given final thrust-to-weight ratio τ_f in each stage , this limiting burnout altitude is obtained by replacing τ_i by τ_f which can be proved.

BURNOUT PERFORMANCE WITH VARYING EQUIVALENT EXHAUST VELOCITY IN EACH STAGE

With the same partial payload ratio $\pi_0^{\frac{1}{n}}$, structural factor ϵ and initial thrust − to − weight ratio τ_i , the equivalent exhaust velocity V_{Ek} in the kth stage is

$$V_{Ek} = V_0 + 2\lambda k \qquad (49)$$

where λ and V_0 are constant and burninig time t_b and burnout velocity V_b are given by

$$t_b = \sum_{k=1}^{n} \frac{V_0 + 2\lambda k}{g\tau_i}(1-\epsilon_k)(1-\pi_k)$$

$$= \frac{n\{V_0 + (n+1)\lambda\}}{g\tau_i}(1-\epsilon)(1-\pi_0^{\frac{1}{n}}) \qquad (50)$$

$$V_b = \sum_{k=1}^{n}(V_0 + 2\lambda k)\left[-\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\} - \frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]$$

$$= n(V_0 + \lambda(n+1))\left[-\log\left\{\epsilon + (1-\epsilon)\pi_0^{\frac{1}{n}}\right\} - \frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right] \qquad (51)$$

Burnout altitude can be computed by cumbersome expressions utilizing $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.But these three parameters unlike previous situations do not admit of limiting values as n→ ∞ ie with infinite number of stages. We see that t_b, $V_b$and $H_b \to \infty$ in this case.

Now we retain the foregoing characteristics for each stage except the equivalent exhaust velocity V_{Ek} which is now considered as

$$V_{Ek} = V_E \lambda^{k-1}, \qquad (k=1,2,3, \ldots n)$$

λ is a constant. Then burning time and burnout velocity are:

$$t_{b_1}=\sum_{k=1}^{n}\frac{V_{Ek}}{g\tau_i}(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)=\sum_{k=1}^{n}\frac{V_E}{g\tau_i}\lambda^{k-1}(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right) \qquad (52)$$

$$=V_E\frac{(\lambda^n-1)(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{(\lambda-1)g\tau_i}$$

$$V_{b_1}=\sum_{k=1}^{n}V_E\lambda^{k-1}\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]$$

$$=V_E\frac{\lambda^n-1}{\lambda-1}\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right] \qquad (53)$$

With tedious calculations the relevant burnout altitude with such stage-wise –variable equivalent exhaust velocity can be determined. In order to find the burnout conditions we find the following limiting values as n tends to infinity ie with infinite number of stages:

$$\lim_{n\to\infty}\lambda^n\left(1-\pi_0^{\frac{1}{n}}\right)_{\lambda>1}=\lim_{n\to\infty}\frac{(1-\pi_0^{\frac{1}{n}})}{(\frac{1}{\lambda})^n}\left(\frac{0}{0}\right)=\lim_{n\to\infty}\frac{\pi_0^{\frac{1}{n}}\log\pi_0}{((\frac{1}{\lambda})^n\log\frac{1}{\lambda})n^2}$$

$$=\frac{\log\pi_0}{\log\frac{1}{\lambda}}\left\{\frac{\lim_{n\to\infty}\pi_0^{\frac{1}{n}}\lambda^n}{n^2}\right\}\left(\frac{\infty}{\infty}\right)=\frac{\log\pi_0}{\log\frac{1}{\lambda}}\lim_{n\to\infty}\lambda^n\frac{\log\lambda}{2n}\left(\frac{\infty}{\infty}\right)=-\frac{\log\pi_0}{2}\lim_{n\to\infty}\lambda^n\log\lambda=\infty \qquad (54)$$

$$\lim_{n\to\infty}\left(1-\pi_0^{\frac{1}{n}}\right)=0 \qquad (55)$$

$$\lim_{n\to\infty}\lambda^n\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]_{\lambda>1}$$

$$=\lim_{n\to\infty}\frac{\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]}{(\frac{1}{\lambda})^n}\left(\frac{0}{0}\right)=\lim_{n\to\infty}\frac{\left[\frac{\pi_0^{\frac{1}{n}}(1-\epsilon)\log\pi_0}{\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}n^2}-\frac{(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0}{n^2\tau_i}\right]}{(\frac{1}{\lambda})^n\log(\frac{1}{\lambda})}\left(\frac{0}{0}\right)$$

$$=-(1-\epsilon)\frac{\log\pi_0}{\log\lambda}\lim_{n\to\infty}\left[\frac{\pi_0^{\frac{1}{n}}}{\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}}-\frac{1}{\tau_i}\right]\lim_{n\to\infty}\frac{\lambda^n}{n^2}\left(\frac{\infty}{\infty}\right)$$

$$=(1-\epsilon)\frac{\log\frac{1}{\pi_0}}{\log\lambda}\left(1-\frac{1}{\tau_i}\right)\mathrm{Lim}_{n\to\infty}\frac{\lambda^n}{2n}\log\lambda\left(\frac{\infty}{\infty}\right)\qquad \lambda>1$$

$$=(1-\epsilon)\frac{\log\frac{1}{\pi_0}}{2\log\lambda}\left(1-\frac{1}{\tau_i}\right)\mathrm{Lim}_{n\to\infty}\lambda^n\log\lambda\left(\frac{\infty}{\infty}\right)=\infty\ ,\ \tau_i>1\qquad(56)$$

Employing equations (54) to (56) in (52) and (53) we see that unlike cases[1] and one previous case,we can show that burning time, burnout velocity and burnout altitude become infinite with infinite number of stages ie as n tends to infinity and can have finite limiting valus if λ tends to 1_-. In that case the limiting burnout velocity turns out to be

$$V_{b_2}=Lim_{\frac{1}{n}\to 0}\frac{V_E(\lambda^n-1)}{(\lambda-1)}\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]$$

Replacing $(1-\lambda)$ by $\frac{1}{n}$ so much so that both tend to 0 as λ tends to 1_- and $\frac{1}{n}\to 0$,

$$V_{b_2}=Lim_{\frac{1}{n}\to 0}\frac{V_E}{\frac{1}{n}}\left[-\log\left\{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}\right\}-\frac{(1-\epsilon)\left(1-\pi_0^{\frac{1}{n}}\right)}{\tau_i}\right]\left(\frac{0}{0}\right)\quad(\lambda^n\to 0)$$

$$=Lim_{\frac{1}{n}\to 0}V_E\left\{\frac{-(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0}{\epsilon+(1-\epsilon)\pi_0^{\frac{1}{n}}}+\frac{(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0}{\tau_i}\right\}==Lim_{\frac{1}{n}\to 0}V_E\left\{-(1-\epsilon)\pi_0^{\frac{1}{n}}\log\pi_0+(1-\epsilon)\pi_0^{\frac{1}{n}}\frac{\log\pi_0}{\tau_i}\right\}$$

$$=V_E(1-\epsilon)\log\left(\frac{1}{\pi_0}\right)\left(1-\frac{1}{\tau_i}\right)\qquad(57)$$

and the limiting burning time is

$$t_{b_2}=\frac{V_E(1-\epsilon)\log\left(\frac{1}{\pi_0}\right)}{g\tau_i}\qquad(58)$$

CONCLUSION

However, in case of given final thrust –to-weight ratio τ_f in each stage the above analysis needs to be replaced by substitution of τ_f in place of τ_i. With given parameters such as burnout velocity or burnout altitude, partial structural factors and overall payload ratio, the minimum burning time to acquire that burnout velocity or burnout altitude and similarly with given other necessary parameters as above, the maximum burnout velocity, maximum burnout altitude or maximum overall payload ratio arise due to equal[1,2] distribution of partial payload ratios among all stages, which can be established by applying optimization technique with Lagrange's multiplier. Further optimum conditions on the preceding lines with the same optimization technique can be obtained by incorporating equal distribution of partial structural ratios among all the stages.

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